20. Reduced homology, excision and applications

Theorem 20.1. Let X be a topological space and let $x_0 \in X$. For all $n \geq 0$,

$$\tilde{H}_n(X) \cong H_n(X, x_0).$$

proof. The proof can be found in most algebraic topology text books. See for example Rotman, Theorem 5.17. ∎

Excision 2: Let X be a topological space and let X_1 and X_2 be subspaces of X with $X = \mathring{X}_1 \cup \mathring{X}_2$. Then the inclusion

$$j : (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2) = (X, X_2)$$

induces isomorphisms

$$j_* : H_n(X_1, X_1 \cap X_2) \to H_n(X, X_2), \quad \forall n \geq 0.$$

Corollary 20.2. (Mayer-Vietoris Theorem for reduced homology).

Let X be a topological space and let X_1, X_2 be subspaces of X with $X = \mathring{X}_1 \cup \mathring{X}_2$, and $X_1 \cap X_2 \neq \emptyset$. Then there is an exact sequence

$$\cdots \to \tilde{H}_n(X_1 \cap X_2) \to \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \to \tilde{H}_n(X) \to \tilde{H}_{n-1}(X_1 \cap X_2) \to \cdots$$

that ends

$$\cdots \to \tilde{H}_0(X_1) \oplus \tilde{H}_0(X_2) \to \tilde{H}_0(X) \to 0.$$

The induced maps are as in Theorem 19.3.

proof. Let $x_0 \in X_1 \cap X_2$. We have the following commutative diagram of inclusions of pairs

$$\begin{tikzcd}
(X_1 \cap X_2, x_0) \arrow[r, "i_1"] \arrow[d, "i_2"'] & (X_1, x_0) \arrow[r, "p"] \arrow[d, "g"] & (X_1, X_1 \cap X_2) \arrow[d, "h"] \\
(X_2, x_0) \arrow[r, "j"'] & (X, x_0) \arrow[r, "q"'] & (X, X_2)
\end{tikzcd}$$

Then the following diagram with exact rows commutes:

$$\begin{tikzcd}
\cdots H_n(X_1 \cap X_2, x_0) \arrow[r, "(i_1)_*"] \arrow[d, "(i_2)_*"'] & H_n(X_1, x_0) \arrow[r, "p_*"] \arrow[d, "g_*"] & H_n(X_1, X_1 \cap X_2) \arrow[r, "d"] \arrow[d, "h_*"] & H_{n-1}(X_1 \cap X_2, x_0) \arrow[r] \arrow[d, "(i_2)_*"] & {} \\
\cdots H_n(X_2, x_0) \arrow[r, "j_*"'] & H_n(X, x_0) \arrow[r, "q_*"'] & H_n(X, X_2) \arrow[r, "\Delta"'] & H_{n-1}(X_2, x_0) \arrow[r] & {}
\end{tikzcd}$$

Excision $\Rightarrow$ h_* isom $\forall n$.

Barratt–Whitehead Theorem (Lemma 19.2) $\Rightarrow$

There is an exact sequence

$$\cdots H_n(X_1 \cap X_2, x_0) \xrightarrow{((i_1)_*, (i_2)_*)} H_n(X_1, x_0) \oplus H_n(X_2, x_0) \xrightarrow{q_* - j_*} H_n(X, x_0) \xrightarrow{D = d h_*^{-1} q_*} H_{n-1}(X_1 \cap X_2, x_0) \to \cdots$$

Theorem 20.1 $\Rightarrow$ there is an exact sequence

$$\cdots \tilde{H}_n(X_1 \cap X_2) \to \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \to \tilde{H}_n(X) \to \tilde{H}_{n-1}(X_1 \cap X_2) \to \cdots$$

$\square$

Homology of Spheres:

Theorem 20.3. Let S^n be the n-sphere, $n \geq 0$. Then

$$H_p(S^0) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } p = 0 \\ 0, & \text{if } p > 0 \end{cases}$$

If $n > 0$, then $H_p(S^n) = \begin{cases} \mathbb{Z}, & \text{if } p = 0 \text{ or } p = n \\ 0, & \text{otherwise} \end{cases}$

Using reduced homology: For every $n \geq 0$,

$$\tilde{H}_p(S^n) = \begin{cases} \mathbb{Z}, & \text{if } p = n \\ 0, & \text{otherwise}. \end{cases}$$

$(\tilde{H}_0(S^0) = \mathbb{Z})$.

<u>Proof.</u> The proof is done by induction on n:

Let $n=0$. Then $S^0 = \{-1,1\}$. Let $x_0 = 1$ and let $i : \{x_0\} \to S^0$ be the inclusion. There is an exact sequence

$$\begin{array}{ccccccc} \cdots \to & H_0(\{x_0\}) & \xrightarrow{i_*} & H_0(S^0) & \xrightarrow{j_*} & H_0(S^0, x_0) & \to 0 \\ & \wr\wr & & \| & & & \\ & \mathbb{Z} & & H_0(\{-1\}) \oplus H_0(\{1\}) & & & \\ & & & \wr\wr \qquad \wr\wr & & & \\ & & & \mathbb{Z} \qquad \mathbb{Z} & & & \end{array}$$

where $i_* : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$, $n \mapsto (0,n)$. Now j_* is a surjection. Thus

$$H_0(S^0, x_0) \cong H_0(S^0)/\ker j_* = H_0(S^0)/\operatorname{im}(i_*) \cong \mathbb{Z}.$$

Theorem 20.1 $\Rightarrow \tilde{H}_0(S^0) \cong H_0(S^0, x_0) \cong \mathbb{Z}$.

path comp.

For $p > 0$, Theorem 20.1 $\Rightarrow \tilde{H}_p(S^0) \cong H_p(S^0, x_0) \cong \underbrace{H_p(\{1\},\{1\})}_{0} \oplus \underbrace{H_p(\{1\},\emptyset)}_{0}$

$= 0$

Assume $n > 0$. Let $\begin{cases} a = \text{the north pole of } S^n \\ b = \text{the south pole of } S^n \end{cases}$

Let $X_1 = S^n - \{a\}$, $X_2 = S^n - \{b\}$. Then $\mathring{X}_1 = X_1$, $\mathring{X}_2 = X_2$ and $S^n = \mathring{X}_1 \cup \mathring{X}_2$, X_1 and X_2 are contractible (homeomorphic to $\mathbb{R}^n$). The intersection $X_1 \cap X_2 = S^n - \{a, b\}$ has the same homotopy type as the equator S^{n-1}. Mayer–Vietoris sequence for reduced homology yields the long exact sequence

$$\cdots \tilde{H}_p(X_1) \oplus \tilde{H}_p(X_2) \to \tilde{H}_p(S^n) \to \tilde{H}_{p-1}(X_1 \cap X_2) \to \tilde{H}_{p-1}(X_1) \oplus \tilde{H}_{p-1}(X_2) \to \cdots$$

with $\tilde{H}_p(X_1) = 0$, $\tilde{H}_p(X_2) = 0$, $\tilde{H}_{p-1}(X_1) = 0$, $\tilde{H}_{p-1}(X_2) = 0$.

Thus $\tilde{H}_p(S^n) \cong \tilde{H}_{p-1}(X_1 \cap X_2) \cong \tilde{H}_{p-1}(S^{n-1}) \overset{\text{induction}}{=} \begin{cases} \mathbb{Z}, & \text{if } p-1 = n-1 \\ 0, & \text{if } p-1 \neq n-1 \end{cases}$

$$= \begin{cases} \mathbb{Z}, & \text{if } p = n \\ 0, & \text{otherwise.} \end{cases}$$

$\square$

<u>Corollary 20.4.</u> Let $n \geq 0$. Then S^n is not a retract of D^{n+1}.

<u>proof.</u> Assume S^n is a retract of D^{n+1}. Then there is a continuous map $r: D^{n+1} \to S^n$ such that $r \circ i: S^n \to S^n$ is the identity, where $i: S^n \hookrightarrow D^{n+1}$ is the inclusion. Then

$$\begin{array}{ccccc} H_n(S^n) & \xrightarrow{i_*} & H_n(D^{n+1}) & \xrightarrow{r_*} & H_n(S^n) \\ \wr\wr & & \wr\wr & & \wr\wr \\ \mathbb{Z} & & 0 & & \mathbb{Z} \end{array}$$

and $r_* \circ i_* = (r \circ i)_* = id_* = id: H_n(S^n) \to H_n(S^n)$, which is impossible since $H_n(D^{n+1}) = 0$. $\square$

<u>Corollary 20.5.</u> (Brouwer's Fixed Point Theorem)

Let $f: D^n \to D^n$ be continuous. Then there is $x \in D^n$ with $f(x) = x$.

<u>proof.</u> Assume $f(x) \neq x$ for all $x \in D^n$. Then the points x and $f(x)$ determine a line. Let $g: D^n \to S^{n-1}$ be the function that assigns to x the point where the ray from $f(x)$ to x intersects S^{n-1}. Then g is continuous and $g(x) = x$ $\forall x \in S^{n-1}$. Thus g is a retraction. Contradiction. $\square$

Corollary 20.6. If $m \neq n$, then S^m and S^n do not have the same homotopy type. In particular, they are not homeomorphic.

proof. A homotopy equivalence would induce an isomorphism $H_p(S^m) \cong H_p(S^n)$, for all $n \geq 0$. $\square$

Corollary 20.7. If $m \neq n$, then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

proof. Assume $f: \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism. Then choose $x_0 \in \mathbb{R}^m$. The restriction

$$f| : \mathbb{R}^m - \{x_0\} \to \mathbb{R}^n - \{f(x_0)\}$$

is a homeomorphism. Contradiction, since $\mathbb{R}^m - \{x_0\}$ has the same homotopy type as S^{m-1} while $\mathbb{R}^n - \{f(x_0)\}$ has the same homotopy type as S^{n-1}, but S^{m-1} and S^{n-1} do not have the same homotopy type. $\square$

Corollary 20.8. If $n \geq 0$, then S^n is not contractible.

proof. If S^n were contractible, it would have the same homology groups as a point. $\square$

Theorem 20.9. Let X be a topological space and let A be a subspace of X. Assume A has a neighbourhood V such that A is a strong deformation retract of V. Let

$$q : (X, A) \to (X/A, A/A)$$

be the quotient map. Then, for all $n \geq 0$, q induces isomorphisms

$$q_* : H_n(X, A) \to H_n(X/A, A/A) \cong \tilde{H}_n(X/A).$$

<u>proof.</u> The following diagram, where the horizontal maps are induced by inclusions, commutes:

$$\begin{array}{ccccc}
H_n(X,A) & \xrightarrow{f} & H_n(X,V) & \xleftarrow{g} & H_n(X-A, V-A) \\
\downarrow q_* & & \downarrow q_* & & \downarrow q_* \\
H_n(X/A, A/A) & \xrightarrow[h]{} & H_n(X/A, V/A) & \xleftarrow[\ell]{} & H_n(X/A - A/A, V/A - A/A)
\end{array}$$

Homomorphism f: A is a strong deformation retract of V $\Rightarrow$ the inclusion $i: A \hookrightarrow V$ is a homotopy equivalence. Thus $i_*: H_n(A) \to H_n(V)$ is an isomorphism, for every n. Consider the exact sequence

$$\to H_n(A) \xrightarrow[\cong]{i_*} H_n(V) \xrightarrow{p} H_n(V,A) \xrightarrow{\Delta} H_{n-1}(A) \xrightarrow[\cong]{i_*} H_{n-1}(V) \to \cdots$$

i_* isom $\Rightarrow$ $im(\Delta) = \ker i_* = 0$ $\Rightarrow$ $\ker(\Delta) = H_n(V,A)$

i_* isom $\Rightarrow$ $\ker p = im(i_*) = H_n(V)$

$\Rightarrow$ $im(p) = 0$.

Exactness $\Rightarrow$ $0 = im(p) = \ker(\Delta) = H_n(V,A)$.

Consider the inclusions $(V,A) \hookrightarrow (X,A) \hookrightarrow (X,V)$.
They induce

$$\begin{array}{ccccc}
S_*(A) & \to & S_*(V) & \to & S_*(V)/S_*(A) \\
\downarrow & & \downarrow & & \downarrow \\
S_*(A) & \to & S_*(X) & \to & S_*(X)/S_*(A) \\
\downarrow & & \downarrow & & \downarrow \\
S_*(V) & \to & S_*(X) & \to & S_*(X)/S_*(V)
\end{array}$$

and a short exact sequence

$$0 \to S_*(V)/S_*(A) \to S_*(X)/S_*(A) \to S_*(X)/S_*(V) \to 0.$$

Hence there is a long exact sequence in homology:

$$\to \cdots \underset{\substack{\| \\ 0}}{H_n(V,A)} \xrightarrow{i_*} H_n(X,A) \xrightarrow{f} H_n(X,V) \xrightarrow{\Delta} \underset{\substack{\| \\ 0}}{H_{n-1}(V,A)} \to H_{n-1}(X,A) \to \cdots$$

$\ker f = \operatorname{im}(i_*) = 0 \Rightarrow f$ is an injection
$\operatorname{im} f = \ker \Delta = H_n(X,V) \Rightarrow f$ is a surjection $\quad \therefore f$ is an isomorphism.

Now, A is a strong deformation retract of V
$\Rightarrow A/A$ is a strong deformation retract of V/A.
Thus, just as we proved that f is an isomorphism, we can show that h is an isomorphism.

Excision I $\Rightarrow g$ and l are isomorphisms.

The quotient map $q: X \to X/A$ restricts to a homeomorphism on the complement of A.
$\Rightarrow$ The map q_* on the right is an isomorphism.
Thus the q_* on the left is

$$(q_*)_{\text{left}} = h^{-1} \circ l \circ (q_*)_{\text{right}} \circ g^{-1} \circ f$$

an isomorphism as a composition of isomorphisms.

$\square$

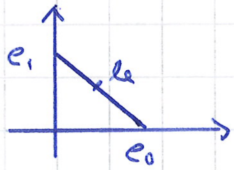
Theorem 20.9 says that we can consider the relative homology groups $H_n(X,A)$ as reduced homology groups of the quotient space X/A when A satisfies the condition of Theorem 20.9.

21. The proof of excision.

Barycentric subdivision of simplices

$\Delta^0 = \{1\}$, can not be divided

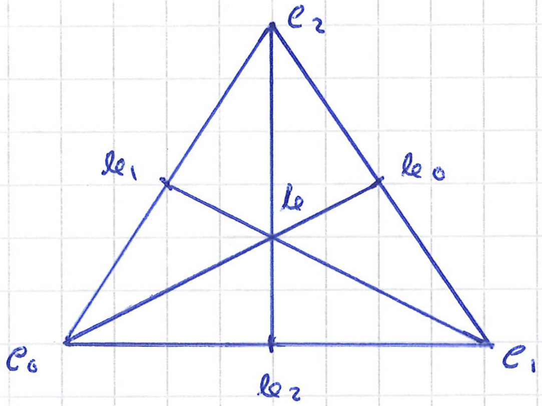
$\Delta^1 = [e_0, e_1]$



b = the midpoint of $[e_0, e_1]$
= the barycenter of $[e_0, e_1]$

Barycentric subdivision of Δ^1 : simplices $[e_0, b]$, $[b, e_1]$

$\Delta^2 = [e_0, e_1, e_2]$



Let : b_0 = the barycenter of $[e_1, e_2]$
b_1 = the barycenter of $[e_0, e_2]$
b_2 = the barycenter of $[e_0, e_1]$
b = the barycenter of $[e_0, e_1, e_2]$

Notice : e_0, e_1, e_2 are barycenters of 0-faces (themselves)
b_1, b_2, b_3 are barycenters of 1-faces
b is the barycenter of $\Delta^2 = [e_0, e_1, e_2]$

Each vertex can be denoted as b^σ, where σ is a face of Δ^2.
For faces τ and σ of Δ^2, write $\tau < \sigma$ if τ is a proper face of σ.

Then $\{b^\tau, b^\sigma, b^\rho\}$ is a triangle exactly when $\tau < \sigma$ and $\sigma < \rho$. $\Rightarrow 3 \cdot 2 = 3!$ triangles.
(or $\tau < \delta < \sigma$
or ...)

Definition 21.1. Let Σ^n be an affine n-simplex. The barycentric subdivision $Sd\,\Sigma^n$ is a family of affine n-simplices defined inductively for $n \geq 0$:

1) $Sd\,\Sigma^0 = \Sigma^0$.
2) Let $\varphi_0, \varphi_1, \dots, \varphi_{n+1}$ be the n-faces of Σ^{n+1} and let b be the barycenter of Σ^{n+1}. Then $Sd\,\Sigma^{n+1}$ consists of all the $(n+1)$-simplices spanned by b and n-simplices $Sd\varphi_i$, $i=0,\dots,n+1$.

Then Σ^n is the union of the n-simplices in $Sd\,\Sigma^n$.

Exercise: Show that $Sd\,\Sigma^n$ consists of exactly $(n+1)!$ n-simplices.

Definition 21.2. Let E be a convex subset of a euclidean space. The barycentric subdivision of E is a homomorphism

$$Sd_n : S_n(E) \to S_n(E)$$

defined inductively on generators $T: \Delta^n \to E$ as follows:

1) If $n=0$, then $Sd_0(T) = T$.
2) If $n>0$, then $Sd_n(T) = T(b_n) \cdot Sd_{n-1}(\partial T)$, where b_n is the barycenter of Δ^n and

$$T(b_n) \cdot Sd_{n-1}(\partial T) = T(b_n) \cdot Sd_{n-1}\Big(\sum_{j=0}^{n} (-1)^j\, T \circ e^j\Big)$$

$$\overset{\text{def}}{=} \sum_{j=0}^{n} (-1)^j\, T(b_n) \cdot Sd_{n-1}(T \circ e^j).$$

Notice: This type of map was already used to calculate the homology groups of a convex subset of $\mathbb{R}^n$, see example 16.4.

Recall: For a convex subset X of $\mathbb{R}^n$, a point $x_0 \in X$ and $T : \Delta^n \to X$ we defined

$$x_0 \cdot T : \Delta^{n+1} \to X$$

by setting

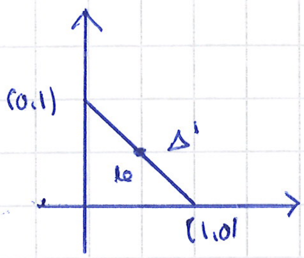
$$(x_0 \cdot T)(t_0, \dots, t_{n+1}) = \begin{cases} x_0, & \text{if } t_0 = 1 \\ t_0 x_0 + (1-t_0)\, T\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right), & \text{if } 0 \le t_0 < 1 \end{cases}$$

Thus, for $\sigma : \Delta^n \to E$,

$$\sigma(b_n) \cdot Sd_{n-1}(\sigma \circ e^j)(t_0, \dots, t_n)$$

$$= \begin{cases} \sigma(b_n), & \text{if } t_0 = 1 \\ t_0 \sigma(b_n) + (1-t_0)\, Sd_{n-1}(\sigma \circ e^j)\left(\frac{t_1}{1-t_0}, \dots, \frac{t_n}{1-t_0}\right), & \text{if } 0 \le t_0 < 1 \end{cases}$$

<u>Example:</u> Let $n = 1$ and let $\sigma = \delta^1 = id : \Delta^1 \to \Delta^1$.
Then $b_n = (\frac{1}{2}, \frac{1}{2})$.



$\partial \delta^1 = \delta^1 \circ e^0 - \delta^1 \circ e^1$, where $\delta^1 \circ e^0(1) = \delta^1(0,1) = (0,1)$
and $\delta^1 \circ e^1(1) = \delta^1(1,0) = (1,0)$

$\delta^1 \circ e^0, \delta^1 \circ e^1 : \Delta^0 \to E \Rightarrow \begin{cases} Sd_0(\delta^1 \circ e^0) = \delta^1 \circ e^0 \\ Sd_0(\delta^1 \circ e^1) = \delta^1 \circ e^1 \end{cases}$

Then $\delta^1(b_n) \cdot Sd_0(\delta^1 \circ e^0)(t_0, t_1)$

$$= \left((\tfrac{1}{2}, \tfrac{1}{2}) \cdot (\delta^1 \circ e^0)\right)(t_0, t_1)$$

$$= \begin{cases} (\frac{1}{2}, \frac{1}{2}), & \text{if } t_0 = 1 \\ t_0(\frac{1}{2}, \frac{1}{2}) + (1-t_0)(\delta^1 \circ e^0)(\underbrace{\tfrac{t_1}{1-t_0}}_{=1}), & \text{if } t_0 \neq 1 \end{cases}$$

$$= \begin{cases} (\frac{1}{2}, \frac{1}{2}), & \text{if } t_0 = 1 \\ t_0(\frac{1}{2}, \frac{1}{2}) + (1-t_0)\underbrace{\delta^1(0,1)}_{(0,1)} = t_0(\frac{1}{2}, \frac{1}{2}) + (1-t_0)(0,1), & \text{if } t_0 \neq 1 \end{cases}$$

Similarly $\delta^1(b_n) \cdot Sd_0(\delta^0 \circ e^1)(t_0, t_1)$

$$= \begin{cases} (\frac{1}{2}, \frac{1}{2}), & \text{if } t_0 = 1 \\ t_0(\frac{1}{2}, \frac{1}{2}) + (1-t_0)\underbrace{(\delta^1 \circ e^1)(1)}_{(1,0)} = t_0(\frac{1}{2}, \frac{1}{2}) + (1-t_0)(1,0), & \text{if } t_0 \neq 1 \end{cases}$$

Let next X be any topological space. The <u>nth barycentric subdivision</u> of X, $n \geq 0$, is the homomorphism

$$Sd_n : S_n(X) \to S_n(X),$$

defined on the generators $\sigma : \Delta_n \to X$ of $S_n(X)$ by

$$Sd_n(\sigma) = \sigma_\# Sd_n(\delta^n),$$

where $\delta^n : \Delta^n \to \Delta^n$ is the identity map.

$$\begin{array}{ccccc} S_n(\Delta^n) & \xrightarrow{Sd_n} & S_n(\Delta^n) & \xrightarrow{\sigma_\#} & S_n(X) \\ \Psi & & \Psi & & \\ \delta^n & \longmapsto & Sd_n(\delta^n) & \longmapsto & \sigma_\# Sd_n(\delta^n) \end{array}$$

<u>Example</u> By the previous example, $Sd_1(\delta^1) = T_0 - T_1$, where

$$T_0 : \Delta^1 \to \Delta^1, \quad T_0(t_0, t_1) = t_0(\tfrac{1}{2}, \tfrac{1}{2}) + (1-t_0)(0,1)$$

$$T_1 : \Delta^1 \to \Delta^1, \quad T_1(t_0, t_1) = t_0(\tfrac{1}{2}, \tfrac{1}{2}) + (1-t_0)(1,0)$$

Let $\sigma : \Delta^1 \to X$. Then

$$Sd_1(\sigma) = \sigma_\# Sd_1(\delta^1) = \sigma_\#(T_0 - T_1) = \sigma \circ T_0 - \sigma \circ T_1 .$$

<u>Lemma 21.3.</u> Let $f : X \to Y$ be continuous. Then the diagram

$$\begin{array}{ccc} S_n(X) & \xrightarrow{Sd_n} & S_n(X) \\ \Big\downarrow f_\# & & \Big\downarrow f_\# \\ S_n(Y) & \xrightarrow[Sd_n]{} & S_n(Y) \end{array}$$

commutes, for every $n \geq 0$.

proof. Let $\sigma : \Delta^n \to X$ be a generator of $S_n(X)$. Then

$$\begin{aligned}(\phi_\# \circ Sd_n)(\sigma) &= \phi_\#(\sigma_\# Sd_n(\delta^n)) \\ &= (\phi_\# \circ \sigma_\#)(Sd_n(\delta^n)) \\ &= (\phi \circ \sigma)_\#(Sd_n(\delta^n)) \\ &= Sd_n(\phi \circ \sigma) \\ &= (Sd_n \circ \phi_\#)(\sigma). \;\square\end{aligned}$$

Lemma 21.9. $Sd : S_*(X) \to S_*(X)$ is a chain map.

proof. First, let's assume X is convex.
Let $\tau : \Delta^n \to X$ be a singular n-simplex.
We show that $Sd_{n-1}\partial_n \tau = \partial_n Sd_n \tau$. The proof is done by induction on n.

Let $n = 0$. Then $S_{-1}(X) = 0$, thus $\partial_0 = 0$ and $Sd_{-1} = 0$.
$\Rightarrow$ $Sd_{-1} \circ \partial_0 = \partial_0 \circ Sd_0$.

Let $n > 0$. Then

$$\begin{aligned}\partial_n Sd_n \tau &= \partial_n(\tau(b_n) \cdot Sd_{n-1}(\partial_n \tau)) \quad \text{(definition of } Sd_n) \\ &= Sd_{n-1}\partial_n \tau - \tau(b_n) \cdot ((\partial_{n-1} Sd_{n-1})\partial_n \tau)\end{aligned}$$

p. 128 : $\partial_0(x_0, \cdot)(\tau) = \tau - (x_0, \cdot)(\partial \tau)$

$$\begin{aligned}&= Sd_{n-1}\partial_n \tau - \tau(b_n) \cdot ((Sd_{n-2}\underbrace{\partial_{n-1})\partial_n}_{=0} \tau) \quad \text{(induction)} \\ &= Sd_{n-1}\partial_n \tau. \;\blacksquare\end{aligned}$$

Let then X be any space, not necessarily convex.
Let $\sigma : \Delta^n \to X$ be a singular n-simplex. Then

$$\begin{aligned}
\partial_n Sd_n(\sigma) &= \partial_n \sigma_\# Sd_n(\delta^n) && \text{(definition of } Sd_n) \\
&= \sigma_\# \partial_n Sd_n(\delta^n) && (\sigma_\# \text{ is a chain map}) \\
&= \sigma_\# Sd_{n-1} \partial_n(\delta^n) && (\Delta^n \text{ is convex}) \\
&= Sd_{n-1} \sigma_\# \partial_n(\delta^n) && \text{(Lemma 21.3.)} \\
&= Sd_{n-1} \partial_n \sigma_\#(\delta^n) && (\sigma_\# \text{ is a chain map}) \\
&= Sd_{n-1} \partial_n(\sigma) && (\sigma_\#(\delta^n) = \sigma)
\end{aligned}$$

□

Lemma 21.5. For each $n \geq 0$, $H_n(Sd): H_n(X) \to H_n(X)$ is the identity.

proof. By Proposition 15.2, it suffices to prove that Sd is chain homotopic to $id: S_*(X) \to S_*(X)$. Therefore, we will construct homomorphisms $T_n: S_n(X) \to S_{n+1}(X)$ satisfying

$$\partial_{n+1} T_n + T_{n-1} \partial_n = id_n - Sd_n.$$

First, let's assume that X is convex, and let's construct the T_n by induction on n. For $n=0$, let

$$T_0 = 0 : S_0(X) \to S_1(X).$$

Then, for a 0-simplex σ,

$$\partial_1 T_0 \sigma = 0 \quad \text{and} \quad id(\sigma) - Sd_0(\sigma) = \sigma - \sigma = 0.$$

Let $n>0$. Then T_n should satisfy

$$\partial_{n+1} T_n \gamma + T_{n-1} \partial_n \gamma = \gamma - Sd_n(\gamma),$$

for any $x \in S_n(X)$. Equivalently, should be

$$\partial_{n+1} T_n x = x - Sd_n(x) - T_{n-1}\partial_n x. \qquad (*)$$

The right side of $(*)$:

$$\partial_n (x - Sd_n(x) - T_{n-1}\partial_n x)$$

$$= \partial_n x - \partial_n Sd_n(x) - \partial_n T_{n-1} \partial_n x$$

$$\overset{\text{by induction}}{=} \partial_n x - \partial_n Sd_n(x) - (id_{n-1} - Sd_{n-1} - T_{n-2}\partial_{n-1})\partial_n x$$

$$= \partial_n x - \partial_n Sd_n(x) - \partial_n x + Sd_{n-1}\partial_n x + T_{n-2}\underbrace{\partial_{n-1}\partial_n}_{=0} x$$

$$= \underbrace{(Sd_{n-1}\partial_n - \partial_n Sd_n)}_{=0,\ \text{since } Sd \text{ is a chain map}}(x) = 0.$$

Thus $x - Sd_n(x) - T_{n-1}\partial_n x \in Z_n(X)$. In the end of Example 16.4, we showed that

$$\partial_{n+1}(b \cdot x) = x,$$

for an $x \in Z_n(X)$. Thus, let's define T_n, by

$$T_n(x) = b \cdot (x - Sd_n(x) - T_{n-1}\partial_n x), \quad b \in X.$$

It follows that

$$\partial_{n+1} T_n(x) = x - Sd_n(x) - T_{n-1}\partial_n x$$
$$= \text{the right side of } (*).$$

We constructed the T_n for convex spaces X.