<u>Induction assumption</u>: Let $n \geq 1$. Assume that

$D_j : S_j(X) \to S_{j+1}(X \times I)$ has been constructed for every $j < n$ in such a way that both the chain homotopy condition and the naturality condition $(*)$ hold.

We next construct $(D_X)_n : S_n(X) \to S_{n+1}(X \times I)$.

Since the condition $(*)$ must hold, the diagram

$$\begin{tikzcd} S_n(X) \arrow[r, "D_X"] \arrow[d, "h_\#"'] & S_{n+1}(X \times I) \arrow[d, "(h \times id)_\#"] \\ S_n(X') \arrow[r, "D_{X'}"'] & S_{n+1}(X' \times I) \end{tikzcd}$$

must commute for every $X'$ and for every $h : X \to X'$.

Choose $X = \Delta_n$ and $h = T : \Delta_n \to X$, so choose $X' = X$.

Then the diagram

$$\begin{tikzcd} S_n(\Delta_n) \arrow[r, "D_{\Delta_n}"] \arrow[d, "T_\#"'] & S_{n+1}(\Delta_n \times I) \arrow[d, "(T \times id)_\#"] \\ S_n(X) \arrow[r, "D_X"'] & S_{n+1}(X \times I) \end{tikzcd} \qquad (*_{\Delta_n})$$

must commute for every $T : \Delta_n \to X$.

Let $id_{\Delta_n} : \Delta_n \to \Delta_n$ be the identity map. Then $id_{\Delta_n} \in S_n(\Delta_n)$

Now, $T_\#(id_{\Delta_n}) = T \circ id_{\Delta_n} = T \in S_n(X)$.

Since the diagram $(*_{\Delta_n})$ commutes, it follows that

$$(T \times id)_\# D_{\Delta_n}(id_{\Delta_n}) = D_X T_\# (id_{\Delta_n}),$$

i.e., $$\underline{\underline{D_X T = D_X T_\#(id_{\Delta_n}) = (T \times id)_\# D_{\Delta_n}(id_{\Delta_n})}}$$

This means that $D_X$ is completely determined when we first construct $D_{\Delta_n}(id_{\Delta_n})$!

$$\begin{array}{ccc} S_n(\Delta_n) & \xrightarrow{D_{\Delta_n}} & S_{n+1}(\Delta_n \times I) \\ \cup & & \\ id_{\Delta_n} & \longmapsto & ? \end{array}$$

Let's consider a certain element $c$ in $S_n(\Delta_n \times I)$:

$$c = (i_1)_\#(id_{\Delta_n}) - (i_0)_\#(id_{\Delta_n}) - D_{\Delta_n}(\partial(id_{\Delta_n}))$$

$$\begin{tikzcd}
 & & & id_{\Delta_n} \arrow[d, phantom, "\cap"] & \partial(id_{\Delta_n}) \arrow[d, phantom, "\cap"] & \\
\cdots \arrow[r] & S_{n+2}(\Delta_n) \arrow[r, "\partial"] \arrow[d] & S_{n+1}(\Delta_n) \arrow[r, "\partial"] \arrow[d] \arrow[dl, "D_{\Delta_n}"'] & S_n(\Delta_n) \arrow[r, "\partial"] \arrow[d] \arrow[dl, "D_{\Delta_n}"'] & S_{n-1}(\Delta_n) \arrow[r] \arrow[d, "(i_0)_\#\ (i_1)_\#"] \arrow[dl, "D_{\Delta_n}"'] & \cdots \quad (\circledcirc) \\
\cdots \arrow[r] & S_{n+2}(\Delta_n \times I) \arrow[r, "\partial"'] & S_{n+1}(\Delta_n \times I) \arrow[r, "\partial"'] & S_n(\Delta_n \times I) \arrow[r, "\partial"'] & S_{n-1}(\Delta_n \times I) \arrow[r] & \cdots \\
 & & & c \arrow[u, phantom, "\cup"] & &
\end{tikzcd}$$

Thus $c \in S_n(\Delta_n \times I)$.

Now,

$$\partial c = \partial (i_1)_\# (id_{\Delta_n}) - \partial (i_0)_\# (id_{\Delta_n}) - \partial D_{\Delta_n}(\partial (id_{\Delta_n}))$$

$$= (i_1)_\# \partial (id_{\Delta_n}) - (i_0)_\# \partial (id_{\Delta_n})$$

$$- \underbrace{\Big[ (i_1)_\# \partial (id_{\Delta_n}) - (i_0)_\# \partial (id_{\Delta_n}) - \overbrace{D_{\Delta_n} \partial (\partial id_{\Delta_n})}^{0,\ \text{since } \partial\partial = 0} \Big]}_{= \partial D_{\Delta_n} (\partial id_{\Delta_n}) \text{ by } (\circledcirc)},$$

$$= 0.$$

Thus $c \in S_n(\Delta_n \times I)$ is such that $\partial c = 0 \in S_{n-1}(\Delta_n \times I)$.

The space $\Delta_n \times I \subset \mathbb{R}^{n+1} \times I \subset \mathbb{R}^{n+2}$ is convex.

Example 16.4 $\Rightarrow$ $H_m(\Delta_n \times I) = 0 \ \forall\ m \geq 1$.

In particular, $H_n(\Delta_n \times I) = 0$.

$\Rightarrow$ $c \in Z_n(\Delta_n \times I) = B_n(\Delta_n \times I)$

$\Rightarrow$ $\exists\ b \in S_{n+1}(\Delta_n \times I) : \partial b = c$.

(i.e. $\partial b + D_{\Delta_n}(\partial (id_{\Delta_n})) = (i_1)_\# (id_{\Delta_n}) - (i_0)_\# (id_{\Delta_n})$ )

$$\begin{tikzcd}
id_{\Delta_n} \in S_n(\Delta_n) \arrow[r, "D_{\Delta_n}"] \arrow[d, "T_\#"] & S_{n+1}(\Delta_n \times I) \ni b \arrow[d, "(T \times id)_\#"] \\
T_\#(id_{\Delta_n}) = T \in S_n(X) \arrow[r, "D_X"'] & S_{n+1}(X \times I)
\end{tikzcd} \qquad (*)$$

$$D_X(T) = (T \times id)_\# D_{\Delta_n}(id_{\Delta_n})$$

Define : $D_{\Delta_n}(id_{\Delta_n}) = b$

Thus, we define $D_X(T) = (T \times id)_\# (b)$

Thus we obtain a homomorphism

$$D_X : S_n(X) \to S_{n+1}(X \times I), \quad D_X\left(\sum n_i T_i\right) = \sum n_i D_X(T_i).$$

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+2}(X) \arrow[r, "\partial"] \arrow[d] & S_{n+1}(X) \arrow[r, "\partial"] \arrow[d] & S_n(X) \arrow[r, "\partial"] \arrow[d] \arrow[dl, "D_X"'] & S_{n-1}(X) \arrow[r] \arrow[d, "(i_0)_\#, (i_1)_\#"] \arrow[dl, "D_X"'] & \cdots \\
\cdots \arrow[r] & S_{n+2}(X \times I) \arrow[r, "\partial"'] & S_{n+1}(X \times I) \arrow[r, "\partial"'] & S_n(X \times I) \arrow[r, "\partial"'] & S_{n-1}(X \times I) \arrow[r] & \cdots
\end{tikzcd}$$

By the induction assumption, $\partial_{j+1}(D_X)_j + (D_X)_{j-1}\partial_j = (i_1)_\# - (i_0)_\#$,
for $j+1 \leq n$
i.e. $j < n$

$j = n$: Let $T : \Delta_n \to X$, $T \times id : \Delta_n \times I \to X \times I$.

Then $\partial D_X(T) + D_X(\partial T)$

$= \partial\left((T \times id)_\#(\ell)\right) + D_X \partial\left(T_\#(id_{\Delta_n})\right)$

$= (T \times id)_\# \underbrace{\partial \ell}_{c} + D_X T_\# \partial(id_{\Delta_n})$

$= (T \times id)_\#(c) + (T \times id)_\# D_{\Delta_n}(\partial id_{\Delta_n})$ $\leftarrow$ (*) on p. 137 commutes for $n-1$ by induction

$= (T \times id)_\#\left((i_1)_\#(id_{\Delta_n}) - (i_0)_\#(id_{\Delta_n}) - D_{\Delta_n}(\partial id_{\Delta_n})\right) + (T \times id)_\# D_{\Delta_n}(\partial id_{\Delta_n})$

$= \underbrace{(T \times id)_\#\left((i_1)_\#(id_{\Delta_n})\right)}_{i_1 T = (T \times id) \circ i_1} - \underbrace{(T \times id)_\#\left((i_0)_\#(id_{\Delta_n})\right)}_{i_0 T = (T \times id) \circ i_0}$

$= (i_1 T)_\#(id_{\Delta_n}) - (i_0 T)_\#(id_{\Delta_n})$

$= (i_1)_\#(T) - (i_0)_\#(T) = \left((i_1)_\# - (i_0)_\#\right)(T).$

$\therefore \partial D_X + D_X \partial = (i_1)_\# - (i_0)_\#.$

Thus $D_X : S_n(X) \to S_{n+1}(X\times I)$ is a chain homotopy between $(i_1)_\#$ and $(i_0)_\#$ up to dimension $n$. We still have to check that the naturality condition holds:

Let $h : X \to X'$. Then

$$
\begin{aligned}
(h\times id)_\# (D_X T) &= (h\times id)_\# (T\times id)_\# (\ell) \\
&= (hT \times id)_\# (\ell) \\
&= D_{X'}(hT) \\
&= D_{X'} h_\# (T),
\end{aligned}
$$

i.e., $(h\times id)_\# D_X = D_{X'} h_\#$

i.e., the diagram

$$
\begin{tikzcd}
S_n(X) \arrow[r, "D_X"] \arrow[d, "h_\#"'] & S_{n+1}(X\times I) \arrow[d, "(h\times id)_\#"] \\
S_n(X') \arrow[r, "D_{X'}"'] & S_{n+1}(X'\times I)
\end{tikzcd}
\qquad (*)
$$

commutes. Thus we are done with the induction step.

We have constructed a chain homotopy $D_X : S_*(X) \to S_*(X\times I)$ between $(i_1)_\#$ and $(i_0)_\#$, satisfying condition $(*)$.

<u>The relative case:</u> Let $A \subset X$ and let $i : A \hookrightarrow X$ be the inclusion. We now have chain homotopies $D_A$ and $D_X$ s.t. the diagram

$$
\begin{tikzcd}
S_n(A) \arrow[r, "D_A"] \arrow[d, "i_\#"'] & S_{n+1}(A\times I) \arrow[d, "(i\times id)_\#"] \\
S_n(X) \arrow[r, "D_X"'] & S_{n+1}(X\times I)
\end{tikzcd}
$$

commutes. Thus $D_X | S_n(A) = D_A \ \forall n$.

Therefore, $D_X$ induces a chain homotopy

$$D(X,A) : S_n(X)/S_n(A) \longrightarrow S_{n+1}(X\times I)/S_{n+1}(A\times I)$$
$$\quad \| \qquad\qquad\qquad\qquad \|$$
$$S_n(X,A) \qquad\qquad S_{n+1}(X\times I, A\times I)$$

between the chain maps

$$(i_0)_\#, (i_1)_\# : S_*(X,A) \longrightarrow S_*(X\times I, A\times I).$$

Assume $f,g : (X,A) \to (Y,B)$ are homotopic, $F : f \simeq g$. Then

$$F_\# D(X,A) : S_*(X,A) \to S_*(Y,B)$$

is a chain homotopy between $f_\#, g_\# : S_*(X,A) \to S_*(Y,B)$, where

$$F_\# : S_*(X\times I, A\times I) \to S_*(Y,B)$$

is the chain map induced by

$$F : (X\times I, A\times I) \to (Y,B). \qquad \square$$

# 18. The Hurewicz Theorem

The Hurewicz Theorem tells how $\pi_1$ and $H_1$ are related.

**Lemma 18.1.** Let $X$ be a topological space. Let $\eta : \Delta^1 \to I$ be the homeomorphism $(1-t)e_0 + te_1 \mapsto t$ (where $e_0 = (1,0)$ and $e_1 = (0,1)$). Then the function

$$\varphi : \pi_1(X, x_0) \mapsto H_1(X),$$
$$[f]_\pi \longmapsto [f\eta]$$

where $f$ is a closed path at $x_0 \in X$, is well defined. (Here $[f]_\pi$ denotes the homotopy class of $f$ while $[f\eta]$ denotes the homology class of $f\eta$.)

**proof.** We have

$$\Delta^1 \xrightarrow{\eta} I \xrightarrow{f} X,$$

thus $f\eta \in S_1(X)$. Now,

$$\partial_1(f\eta) = f\eta(0,1) - f\eta(1,0) = f(1) - f(0) = 0.$$

$\Rightarrow f\eta \in Z_1(X)$, and $[f\eta] \in H_1(X)$.

Let $u : I \to S^1$, $t \mapsto e^{i2\pi t}$. Then $u\eta \in Z_1(S^1)$.
Let $f' : S^1 \to X$, $f'(e^{i2\pi t}) = f(t)$.
Then the diagram

$$\begin{tikzcd} I \arrow[r, "u"] \arrow[dr, "f"'] & S^1 \arrow[d, "f'"] \\ & X \end{tikzcd}$$

commutes.

Thus $f' : S^1 \to X$ induces a chain map $f'_\# : S_*(S^1) \to S_*(X)$, and, in particular, a homomorphism

$$f'_* : H_1(S^1) \to H_1(X), \quad [c] \mapsto [f'_\#(c)].$$

Then

$$[f\gamma] = [f' \circ \gamma] = f'_* \underbrace{[\circ\gamma]}_{\in H_1(S^1)} \in H_1(X).$$

Assume $g : I \to X$ is a closed path and $F : f \simeq g$ rel $\dot{I}$.

Then $F' : f' \simeq g'$, where

$$F' : S^1 \times I \to X, \quad (e^{i2\pi t}, s) \mapsto F(t, s).$$

$$(F'(e^{i2\pi t}, 0) = F(t, 0) = f(t) = f'(e^{2\pi i t}), \ldots)$$

Homotopy axiom (Thm 17.2) $\Rightarrow$

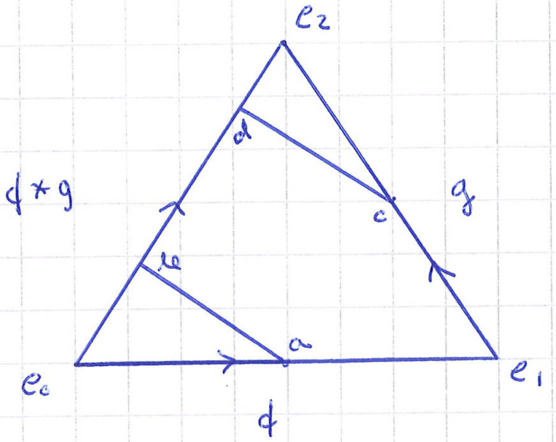
$$[f\gamma] = f'_*[\circ\gamma] = g'_*[\circ\gamma] = [g\gamma]. \qquad \square$$

The map $\varphi : \pi_1(X, x_0) \to H_1(X)$ is called the <u>Hurewicz map.</u>

<u>Theorem 18.2.</u> The Hurewicz map $\varphi : \pi_1(X, x_0) \to H_1(X)$ is a homomorphism.

<u>proof.</u> Let $f, g : I \to X$ be closed paths at $x_0$. We use the following triangle to define a continuous map

$$\sigma : \Delta^2 \to X.$$




On the boundary of $\Delta^2$:

$$\sigma(1-t,t,0)=f(t)$$
$$\sigma(0,1-t,t)=g(t)$$
$$\sigma(1-t,0,t)=(f*g)(t)$$

On the entire $\Delta^2$:

Require $\sigma$ to be constant on the following line segments:

1) on the segments with endpoints
$a=a(t)=(1-t,t,0)$ and
$b=b(t)=\left(\frac{2-t}{2},0,\frac{t}{2}\right)$,
where $0\le t\le 1$
(Here $\sigma\left(\frac{2-t}{2},0,\frac{t}{2}\right)=(f*g)\left(\frac{t}{2}\right)=f\left(2\cdot\frac{t}{2}\right)=f(t)=\sigma(1-t,t,0)$.)

2) on the segments with endpoints
$c=c(t)=(0,1-t,t)$ and
$d=d(t)=\left(\frac{1-t}{2},0,\frac{1+t}{2}\right)$ where $0\le t\le 1$
(Here $\sigma\left(\frac{1-t}{2},0,\frac{1+t}{2}\right)=(f*g)\left(\frac{1+t}{2}\right)=g\left(2\cdot\frac{1+t}{2}-1\right)=g(t)=\sigma(0,1-t,t)$.)

Then $\sigma:\Delta^2\to X$ is continuous. $\Rightarrow \sigma\in S_2(X)$.

Now,
$$\Delta^1 \xrightarrow{e^0,e^1,e^2} \Delta^2 \xrightarrow{\sigma} X \qquad \sigma(t_0,t_1,t_2)$$

$$(\sigma\circ e^0)(1-t,t)=\sigma(0,1-t,t)=g(t)=(g\circ\eta)(1-t,t) \qquad (\eta(1-t,t)=t)$$
$$(\sigma\circ e^1)(1-t,t)=\sigma(1-t,0,t)=(f*g)(t)=(f*g)\circ\eta(1-t,t)$$
$$(\sigma\circ e^2)(1-t,t)=\sigma(1-t,t,0)=f(t)=(f\circ\eta)(1-t,t).$$

Thus $\partial\sigma = \sigma\circ e^0 - \sigma\circ e^1 + \sigma\circ e^2$
$= g\circ\gamma - (f * g)\circ\gamma + f\circ\gamma.$

Therefore,

$$\varphi([f]_\pi [g]_\pi) = \varphi([f*g]_\pi) = [(f*g)\circ\gamma]$$

$$= [g\circ\gamma + f\circ\gamma - \partial\sigma]$$

$$= [g\circ\gamma] + [f\circ\gamma] = [f\circ\gamma] + [g\circ\gamma].$$

$\square$

__Theorem 18.3.__ Let $X$ be a path connected topological space. Then the Hurewicz map $\varphi : \pi_1(X, x_0) \to H_1(X)$ is a surjection, the kernel of $\varphi$ is the __commutator subgroup__ of $\pi_1(X, x_0)$, denoted by $\pi_1'(X, x_0)$. Thus there is an isomorphism

$$\tilde{\varphi} : \pi_1(X, x_0)/\pi_1'(X, x_0) \to H_1(X).$$

~~proof.~~ Rotman, Thm 4.29. (This theorem is called Hurewicz theorem, the proof can be found in most algebraic topology textbooks.) $\square$

__Commutator subgroup:__

__Definition 18.4.__ Let $G$ be a group. The subgroup of $G$ generated by the set $\{aba^{-1}b^{-1} \mid a, b \in G\}$ is called the commutator subgroup of $G$ and denoted by $G'$.

__Theorem 18.5.__ Let $G$ be a group. Then $G'$ is a normal subgroup of $G$ and $G/G'$ is abelian.

proof: Let $f: G \to G$ be any isomorphism. Then

$$f(aba^{-1}b^{-1}) = f(a)f(b)f(a)^{-1}f(b)^{-1} \in G'.$$

Thus $f(G') \subset G'$. Let $a \in G$ and let

$$f_a : G \to G, \; g \mapsto aga^{-1}.$$

Then $f_a$ is an isomorphism. Thus

$$aG'a^{-1} = f_a(G') \subset G'.$$

Similarly, $a^{-1}G'a = f_{a^{-1}}(G') \subset G'$ $\Rightarrow G' \subset aG'a^{-1}$.

Thus $aG'a^{-1} = G'$. Since this holds $\forall a \in G$, it follows that $G'$ is a normal subgroup of $G$.
For every $a, b \in G$,

$$(ab)(ba)^{-1} = aba^{-1}b^{-1} \in G'$$

$$\Rightarrow abG' = baG'$$

Thus $G/G'$ is abelian. $\square$

Theorem 18.3 implies the following:

Corollary 18.4. $H_1(S^1) \cong \mathbb{Z}$. $\square$

Corollary 18.5. If $X$ is simply connected, then $H_1(X) = 0$. $\square$

## Reduced homology:

Let $X$ be a topological space.
The singular chain complex of $X$ is

$$\rightarrow S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} S_{n-2}(X) \rightarrow \dots \rightarrow S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0$$

For the reduced homology $\tilde{H}_*(X)$ we change $S_{-1}(X)$ and $\partial_0$:

$$\rightarrow S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0'} \mathbb{Z}, \qquad (*)$$

where $\partial_0'\left(\sum_x n_x x\right) = \sum_x n_x$ (finite sums)

Let $T: \Delta_1 \rightarrow X$.
Then
$$\partial_1 T = T \circ e^0 - T \circ e^1 \in S_0(X)$$
and
$$\partial_0'(\partial_1 T) = 1 + (-1) = 0 \quad \Rightarrow \quad \partial_0' \circ \partial_1 = 0$$

Thus $(*)$ is a chain complex.

The reduced homology groups $\tilde{H}_*(X)$ of $X$ are

$$\tilde{H}_n(X) = H_n(X), \quad n > 0$$
$$\tilde{H}_0(X) = \ker \partial_0' / \operatorname{im} \partial_1$$

**Notice:** We already had the homomorphism $\partial_0' : S_0(X) \rightarrow \mathbb{Z}$ in Proposition 13.3 ($H_0(X) \cong \mathbb{Z}$ for pathconnected $X$). We then used notation $\eta$ for $\partial_0'$ and proved that $\ker \eta = \operatorname{im}(\partial_1(S_1)) = B_0(X)$

Thus $\tilde{H}_0(X) = \ker \partial_0' / \operatorname{im} \partial_1 = 0$ for pathconnected $X$.

# 19. Excision

$X$ = topological space, $A \subset X$
$\Rightarrow$ $\bar{A}$ = the closure of $A$
$\mathring{A}$ = the interior of $A$

**Excision 1.** Let $X$ be a topological space and let $U \subset A \subset X$ be subspaces with $\bar{U} \subset \mathring{A}$.
Then the inclusion $i : (X-U, A-U) \hookrightarrow (X, A)$ induces isomorphisms

$$i_* : H_n(X-U, A-U) \to H_n(X, A)$$

for every $n$.

**Excision 2.** Let $X$ be a topological space and let $X_1$ and $X_2$ be subspaces of $X$ with $X = \mathring{X}_1 \cup \mathring{X}_2$. Then the inclusion

$$j : (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2) = (X, X_2)$$

induces isomorphisms

$$j_* : H_n(X_1, X_1 \cap X_2) \to H_n(X, X_2)$$

for every $n$.

**Theorem 19.1.** Excision 1 is equivalent to Excision 2.

**proof.**

1) Assume Excision 1 holds. Let $X_1, X_2$ be subspaces of $X$ with $X = \mathring{X}_1 \cup \mathring{X}_2$. Let $A = X_2$, $U = X - X_1$.

Show: $\bar{U} \subset \mathring{A}$: $\mathring{X}_1 \subset X_1 \Rightarrow X - X_1 \subset X - \mathring{X}_1$.
$\Rightarrow \bar{U} = \overline{X - X_1} \subset X - \mathring{X}_1$ $\leftarrow$ closed set

Now,

$$X - \mathring{X}_1 = (\mathring{X}_1 \cup \mathring{X}_2) - \mathring{X}_1 = \mathring{X}_2 - \mathring{X}_1$$
$$\subset \mathring{X}_2 = \mathring{A}$$

$\therefore \bar{U} \subset \mathring{A}$

Also, $\quad X - U = X - (X - X_1) = X_1$

and $\quad A - U = X_2 - (X - X_1) = X_2 \cap X_1 = X_1 \cap X_2$

Thus $\quad (X-U, A-U) = (X_1, X_1 \cap X_2)$

and $\quad (X, A) = (X, X_2)$.

By Excision 1, the inclusion

$$(X_1, X_1 \cap X_2) = (X-U, A-U) \hookrightarrow (X, A) = (X, X_2)$$

induces isomorphisms

$$H_n(X_1, X_1 \cap X_2) \to H_n(X, X_2), \quad \text{for every } n.$$

Assume then that Excision 2 holds. Let $U$ and $A$ be subspaces of $X$ such that $\bar{U} \subset \mathring{A}$. Define

$$X_1 = X - U \quad \text{and} \quad X_2 = A.$$

Since $U \subset \bar{U} \subset \mathring{A}$, it follows that

$$X - \mathring{A} \subset X - \bar{U} \subset X - U.$$

Since $X - \bar{U}$ is open,

$$X - \mathring{A} \subset X - \bar{U} = (X - \bar{U})^\circ.$$

Thus

$$\begin{aligned} X = (X - \mathring{A}) \cup \mathring{A} &\subset (X - \bar{U})^\circ \cup \mathring{A} \\ &\subset (X - U)^\circ \cup \mathring{A} = \mathring{X}_1 \cup \mathring{X}_2. \end{aligned}$$

Then

$$(X_1, X_1 \cap X_2) = (X-U, A-U) \quad \text{and} \quad (X, X_2) = (X, A)$$

and the inclusion

$$(X-U, A-U) = (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2) = (X, A)$$

induces isomorphisms

$$H_n(X-U, A-U) \to H_n(X, A), \quad \text{for every } n. \ \square$$

## Lemma 19.2 (Barratt-Whitehead)

Consider the following commutative diagram of abelian groups and homomorphisms

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"] & B_n \arrow[r, "p_n"] \arrow[d, "g_n"] & C_n \arrow[r, "d_n"] \arrow[d, "h_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & A'_n \arrow[r, "j_n"'] & B'_n \arrow[r, "q_n"'] & C'_n \arrow[r, "\Delta_n"'] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where the rows are exact sequences and the maps $h_n$ are isomorphisms. Then there is an exact sequence

$$\cdots \longrightarrow A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - j_n} B'_n \xrightarrow{d_n h_n^{-1} q_n} A_{n-1} \longrightarrow \cdots,$$

where $(i_n, f_n)(a_n) = (i_n(a_n), f_n(a_n))$
and $(g_n - j_n)(b_n, a'_n) = g_n(b_n) - j_n(a'_n)$.

proof HW.

## Theorem 19.3 (Mayer-Vietoris)

Let $X$ be a topological space and let $X_1, X_2$ be subspaces of $X$ with $X = \mathring{X}_1 \cup \mathring{X}_2$. Then there is an exact sequence

$$\cdots \longrightarrow H_n(X_1 \cap X_2) \xrightarrow{(i_{1*}, i_{2*})} H_n(X_1) \oplus H_n(X_2) \xrightarrow{g_* - j_*} H_n(X) \xrightarrow{D} H_{n-1}(X_1 \cap X_2) \longrightarrow \cdots$$

Here:

$$\left.\begin{array}{l} i_1 : X_1 \cap X_2 \hookrightarrow X_1 \\ i_2 : X_1 \cap X_2 \hookrightarrow X_2 \\ g : X_1 \hookrightarrow X \\ j : X_2 \hookrightarrow X \end{array}\right\} \text{inclusions}$$

$$\left.\begin{array}{l} q : (X, \emptyset) \hookrightarrow (X, X_2) \\ h : (X_1, X_1 \cap X_2) \hookrightarrow (X, X_2) \end{array}\right\} \text{inclusions}$$

$d : H_n(X_1, X_1 \cap X_2) \to H_{n-1}(X_1 \cap X_2)$ connecting homom

$D = d h_*^{-1} q_*$

<u>proof.</u> The following diagram (all maps inclusions) commutes:

$$\begin{tikzcd}
(X_1 \cap X_2, \emptyset) \arrow[r, "i_1"] \arrow[d, "i_2"'] & (X_1, \emptyset) \arrow[r, "p"] \arrow[d, "g"] & (X_1, X_1 \cap X_2) \arrow[d, "h"] \\
(X_2, \emptyset) \arrow[r, "j"'] & (X, \emptyset) \arrow[r, "q"'] & (X, X_2)
\end{tikzcd}$$

This diagram induces a commutative diagram of chain complexes:

$$\begin{tikzcd}
0 \arrow[r] & S_*(X_1 \cap X_2) \arrow[r, "(i_1)_\#"] \arrow[d, "(i_2)_\#"'] & S_*(X_1) \arrow[r, "p_\#"] \arrow[d, "g_\#"] & S_*(X_1, X_1 \cap X_2) \arrow[r] \arrow[d, "h_\#"] & 0 \\
0 \arrow[r] & S_*(X_2) \arrow[r, "j_\#"'] & S_*(X) \arrow[r, "q_\#"'] & S_*(X, X_2) \arrow[r] & 0
\end{tikzcd}$$

where the horizontal lines are exact. By Corollary 14.13, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X_1 \cap X_2) \arrow[r, "(i_1)_*"] \arrow[d, "(i_2)_*"'] & H_n(X_1) \arrow[r, "p_*"] \arrow[d, "g_*"] & H_n(X_1, X_1 \cap X_2) \arrow[r, "d"] \arrow[d, "h_*"] & H_{n-1}(X_1 \cap X_2) \arrow[r] \arrow[d, "(i_2)_*"] & \cdots \\
\cdots \arrow[r] & H_n(X_2) \arrow[r, "j_*"'] & H_n(X) \arrow[r, "q_*"'] & H_n(X, X_2) \arrow[r, "\Delta"'] & H_{n-1}(X_2) \arrow[r] & \cdots
\end{tikzcd}$$

commutes and the horizontal lines are exact. ($d$ and $\Delta$ are connecting homomorphisms) Excision 2 $\Rightarrow$ the homomorphisms $h_*$ are isomorphisms. By Lemma 19.2, there is an exact sequence

$$\cdots \to H_n(X_1 \cap X_2) \xrightarrow{((i_1)_*, (i_2)_*)} H_n(X_1) \oplus H_n(X_2) \xrightarrow{g_* - j_*} H_n(X) \xrightarrow{D = d h_*^{-1} q_*} H_{n-1}(X_1 \cap X_2) \to \cdots$$

$\square$