Notice: $S_n(X) = \sum_{T \in Map(\Delta_n, X)} \oplus \mathbb{Z}_T = \underbrace{\sum_{T \in Map_1} \oplus \mathbb{Z}_T}_{S_n(A)} \oplus \sum_{T \in Map_2} \oplus \mathbb{Z}_T$,

where $Map_1 = \{ T \in Map(\Delta_n, X) \mid T(\Delta_n) \subset A \}$

and $Map_2 = \{ T \in Map(\Delta_n, X) \mid T(\Delta_n) \cap (X - A) \neq \emptyset \}$.

Thus $S_n(X)/S_n(A) \cong \sum_{T \in Map_2} \oplus \mathbb{Z}_T$,

i.e., $S_n(X)/S_n(A)$ is a free abelian group for every n.

Definition 16.1. The n^{TH} relative homology group $H_n(X,A)$ of the pair (X,A) is the n^{TH} homology group $H_n(S_*(X)/S_*(A))$ of the quotient complex $S_*(X)/S_*(A)$.

According to Lemma 14.11, the short exact sequence

$$0 \longrightarrow S_*(A) \xrightarrow{i_\#} S_*(X) \xrightarrow{j} S_*(X)/S_*(A) \longrightarrow 0$$

induces a long exact sequence in homology:

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\Delta} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\Delta} H_{n-1}(A) \to \cdots$$

This sequence is called the exact homology sequence of the pair (X,A). (E-S Axiom 4)

Let (X,A) and (Y,B) be topological pairs and let $f:(X,A) \to (Y,B)$ be a continuous map. Thus $f: X \to Y$ is continuous and $f(A) \subset B$. Let $f|: A \to B$ denote the restriction

Then ϕ induces the map

$$\phi_\# : S_*(X) \to S_*(Y)$$

and $\phi_\#(S_n(A)) \subset S_n(B)$, for every n.

Therefore we obtain

$$\phi_\# : S_n(X)/S_n(A) \to S_n(Y)/S_n(B), \text{ for every } n.$$

For every n, the diagram

$$\begin{array}{ccccc} S_n(A) & \xrightarrow{i_*} & S_n(X) & \xrightarrow{j_*} & S_n(X)/S_n(A) \\ \downarrow \phi_\# & & \downarrow \phi_\# & & \downarrow \phi_\# \\ S_n(B) & \xrightarrow[i_*]{} & S_n(Y) & \xrightarrow[j_*]{} & S_n(Y)/S_n(B) \end{array}$$

commutes.

The chain map $\phi_\# : S_n(X)/S_n(A) \to S_n(Y)/S_n(B)$ induces

$$\phi_* : H_n(X,A) \to H_n(Y,B), \text{ for every } n.$$

Let $g : (Y,B) \to (Z,C)$ be a continuous function.
Then

$$(g \circ \phi)_* = g_* \circ \phi_* : H_n(X,A) \to H_n(Z,C)$$

and

$$id : (X,A) \to (X,A)$$

induces

$$id_* = id : H_n(X,A) \to H_n(X,A), \text{ for every } n.$$

<u>$A = \emptyset$:</u> $S_n(\emptyset) = \{0\} \quad \Big(= \sum_{T \in Map(\Delta_n,\emptyset) = \emptyset} \oplus \mathbb{Z}_T \Big) \quad \forall n$

$$\Rightarrow S_n(X)/S_n(\emptyset) = S_n(X)/\{0\} = S_n(X)$$

$$H_n(X,\emptyset) = H_n(S(X)/S(\emptyset)) = H_n(X) \quad \forall n$$

A continuous function $f: (X, A) \to (Y, B)$ induces

$$\begin{tikzcd}
0 \arrow[r] & S(A) \arrow[r, "i"] \arrow[d, "f_\#"] & S(X) \arrow[r, "j"] \arrow[d, "f_\#"] & S(X)/S(A) \arrow[r] \arrow[d, "f_\#"] & 0 \\
0 \arrow[r] & S(B) \arrow[r, "i'"] & S(Y) \arrow[r, "j'"] & S(Y)/S(B) \arrow[r] & 0
\end{tikzcd} \qquad (**)$$

where the horizontal lines are exact sequences.

By Corollary 14.13, the diagram $(\square)$

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(X,A) \arrow[r, "\Delta"] \arrow[d] & H_n(A) \arrow[r, "i_*"] \arrow[d] & H_n(X) \arrow[r, "j_*"] \arrow[d] & H_n(X,A) \arrow[r, "\Delta"] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & {} \\
\cdots \arrow[r] & H_{n+1}(Y,B) \arrow[r, "\Delta'"'] & H_n(B) \arrow[r, "i'_*"'] & H_n(Y) \arrow[r, "j'_*"'] & H_n(Y,B) \arrow[r, "\Delta'"'] & H_{n-1}(B) \arrow[r] & {}
\end{tikzcd}$$

commutes and the horizontal lines are long exact sequences. Since the squares having Δ and Δ' are commutative, it follows that singular homology theory satisfies Eilenberg–Steenrod axiom 3.

<u>Proposition 16.2.</u> Let $f: (X, A) \to (Y, B)$ be a continuous function. Assume

$$(f|A)_*: H_n(A) \to H_n(B) \quad \text{and} \quad f_*: H_n(X) \to H_n(Y)$$

are isomorphisms for every n. Then also

$$f_*: H_n(X, A) \to H_n(Y, B)$$

is an isomorphism for every n.

<u>proof.</u> The proof follows immediately from the fact that the diagram $(\square)$ commutes and from the <u>5-lemma.</u> $\square$

__Lemma 16.3. (5-Lemma)__ Let

$$\begin{tikzcd}
C_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"'] & C_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"'] & C_3 \arrow[r, "\alpha_3"] \arrow[d, "\varphi_3"'] & C_4 \arrow[r, "\alpha_4"] \arrow[d, "\varphi_4"'] & C_5 \arrow[d, "\varphi_5"'] \\
D_1 \arrow[r, "\beta_1"'] & D_2 \arrow[r, "\beta_2"'] & D_3 \arrow[r, "\beta_3"'] & D_4 \arrow[r, "\beta_4"'] & D_5
\end{tikzcd}$$

be a commutative diagram of abelian groups and homomorphisms. Assume the horizontal lines are exact. (This means that the top line is exact at C_2, C_3 and C_4, and the bottom line is exact at D_2, D_3 and D_4). Then the following holds:

1) If φ_2 and φ_4 are surjective and φ_5 is injective, then φ_3 is surjective.

2) If φ_2 and φ_4 are injective and φ_1 is surjective, then φ_3 is injective.

In particular: If $\varphi_1, \varphi_2, \varphi_3$ and φ_4 are isomorphisms, then also φ_3 is an isomorphism.

__Proof.__

1) Assume φ_2 and φ_4 are surjective and φ_5 is injective. We show that φ_3 is surjective. Let $d_3 \in D_3$.
Then $\beta_3(d_3) \in D_4$.
φ_4 surj $\Rightarrow \exists c_4 \in C_4 : \varphi_4(c_4) = \beta_3(d_3)$ $\quad (*)$
Then $\varphi_5(\alpha_4(c_4)) = \beta_4(\varphi_4(c_4)) = \beta_4(\beta_3(d_3)) = 0$, since $\beta_4 \circ \beta_3 = 0$.
φ_5 inj $\Rightarrow \alpha_4(c_4) = 0$,
$\Rightarrow c_4 \in \ker \alpha_4 = \operatorname{im} \alpha_3$
$\Rightarrow \exists c_3 \in C_3 : \alpha_3(c_3) = c_4$.
Then
$$\begin{aligned}\beta_3(\varphi_3(c_3) - d_3) &= \beta_3(\varphi_3(c_3)) - \beta_3(d_3) \\ &= \varphi_4(\alpha_3(c_3)) - \beta_3(d_3) \\ &= \varphi_4(c_4) - \beta_3(d_3) = 0 \quad \text{by } (*)\end{aligned}$$

$\Rightarrow f_3(c_3) - d_3 \in \ker\beta_3 = \operatorname{im}\beta_2$

$\Rightarrow \exists\, d_2 \in D_2 : \beta_2(d_2) = f_3(c_3) - d_3.$

f_2 surj. $\Rightarrow \exists\, c_2 \in C_2 : f_2(c_2) = d_2.$

Then $f_3(\alpha_2(c_2)) = \beta_2(f_2(c_2)) = \beta_2(d_2) = f_3(c_3) - d_3.$

$\Rightarrow d_3 = f_3(c_3) - f_3(\alpha_2(c_2)) = f_3(c_3 - \alpha_2(c_2)).$

$\therefore d_3 \in \operatorname{im} f_3$

$\therefore f_3$ is surjective

2) Assume then that f_2 and f_4 are injective and f_1 is surjective. We show that f_3 is injective.

Let $c_3 \in C_3$. Assume $f_3(c_3) = 0$.

Then $f_4\alpha_3(c_3) - \beta_3\underbrace{f_3(c_3)}_{=0} = 0.$

f_4 inj. $\Rightarrow \alpha_3(c_3) = 0.$

$\Rightarrow c_3 \in \ker\alpha_3 = \operatorname{im}\alpha_2.$

$\Rightarrow \exists\, c_2 \in C_2 : \alpha_2(c_2) = c_3.$

$\Rightarrow \beta_2 f_2(c_2) = f_3\alpha_2(c_2) = f_3(c_3) = 0.$

$\Rightarrow f_2(c_2) \in \ker\beta_2 = \operatorname{im}\beta_1.$

$\Rightarrow \exists\, d_1 \in D_1 : \beta_1(d_1) = f_2(c_2).$

f_1 surj. $\Rightarrow \exists\, c_1 \in C_1 : f_1(c_1) = d_1.$

Then $f_2\alpha_1(c_1) = \beta_1 f_1(c_1) = \beta_1(d_1) = f_2(c_2).$

f_2 inj $\Rightarrow \alpha_1(c_1) = c_2.$

Since $\alpha_1(c_1) = c_2$ and $\alpha_2(c_2) = c_3$, it follows that

$0 = \underbrace{\alpha_2\alpha_1(c_1)}_{=0} = \alpha_2(c_2) = c_3.$

$\therefore f_3$ is injective.

$\square$

Example 16.4. Let $X \subset \mathbb{R}^n$ be convex and $\neq \emptyset$. Then

$$\begin{cases} H_0(X) \cong \mathbb{Z} \\ H_n(X) \cong 0 \quad \forall n > 0 \end{cases}$$

proof. Recall: $X \subset \mathbb{R}^n$ is convex, if and only if

$$\{(1-t)x + ty \mid x, y \in X, \ 0 \le t \le 1\} \subset X.$$

Now, X convex $\Rightarrow$ X path connected

$$\overset{\text{Prop 13.3}}{\Longrightarrow} H_0(X) \cong \mathbb{Z}.$$

Let $x_0 \in X$.
For $T \in S_n(X)$, i.e., for $T : \Delta_n \to X$, define

$$x_0 \cdot T : \Delta_{n+1} \to X,$$

$$(x_0 \cdot T)(t_0, \dots, t_{n+1}) = \begin{cases} x_0, & \text{if } t_0 = 1 \\ t_0 x_0 + (1-t_0)\, T\big(\underbrace{\tfrac{t_1}{1-t_0}, \dots, \tfrac{t_{n+1}}{1-t_0}}_{\in \Delta_n, \text{ since}}\big), & \text{if } 0 \le t_0 < 1 \end{cases}$$

$$\sum_{i=1}^{n+1} \frac{t_i}{1-t_0} = \frac{1}{1-t_0} \sum_{i=1}^{n+1} t_i = \frac{1-t_0}{1-t_0} = 1.$$

Now, $\left.\begin{array}{l} x_0 \in X \\ T\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right) \in X \end{array}\right\}$

$\Rightarrow$ $t_0 x_0 + (1-t_0)\, T\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right) \in X$ since X is convex

$\therefore$ $x_0 \cdot T$ is well defined

We next check that $x_0 \cdot T$ is continuous:

Clearly, $x_0 \cdot T$ is continuous at points $(t_0, \dots, t_{n+1})$, where $t_0 \neq 1$.

$x_0 \cdot T$ is continuous at $(1,0,\dots,0)$:

Notice $(x_0 \cdot T)(1,0,\dots,0) = x_0$

For $t_0 < 1$,

$$\|(x_0 \cdot T)(t_0,\dots,t_{n+1}) - x_0\| = \left\|(1-t_0)T\left(\frac{t_1}{1-t_0},\dots,\frac{t_{n+1}}{1-t_0}\right) + (t_0-1)x_0\right\|$$

$$\leq (1-t_0)\left(\left\|T\left(\frac{t_1}{1-t_0},\dots,\frac{t_{n+1}}{1-t_0}\right)\right\| + \|x_0\|\right)$$

Here: Δ_n compact $\Rightarrow T(\Delta_n)$ compact (T is continuous)

$\Rightarrow T(\Delta_n)$ bounded

$$\Rightarrow \exists M>0 : \left\|T\left(\frac{t_1}{1-t_0},\dots,\frac{t_{n+1}}{1-t_0}\right)\right\| + \|x_0\| \leq M \quad \forall \left(\frac{t_1}{1-t_0},\dots,\frac{t_{n+1}}{1-t_0}\right) \in \Delta_n.$$

$$\Rightarrow (1-t_0)\Bigg(\underbrace{\left\|T\left(\frac{t_1}{1-t_0},\dots,\frac{t_{n+1}}{1-t_0}\right)\right\| + \|x_0\|}_{<M}\Bigg) \to 0 \text{ when } t_0 \to 1$$

$\therefore$ $x_0 \cdot T$ is continuous at $(1,0,\dots,0)$

$\therefore$ $x_0 \cdot T$ is continuous

Define $(x_0,\cdot) : S_n(X) \to S_{n+1}(X)$, $(x_0,\cdot)(T) = x_0 \cdot T$

(i.e., $(x_0,\cdot)\left(\sum_{i=1}^{k} n_i T_i\right) = \sum_{i=1}^{k} n_i (x_0 \cdot T_i)$)

Then $(x_0,\cdot)$ is a homomorphism.

<u>Determine the boundary $\partial(x_0 \cdot T)$:</u>

$((x_0 \cdot T)\circ e_j) : \Delta_n \to X$

$((x_0 \cdot T)\circ e_j)(t_0, t_1,\dots,t_n) = (x_0 \cdot T)(t_0,\dots,t_{j-1},0,t_j,\dots,t_n)$, $0 \leq j \leq n+1$

Then : $n=0, j=0$: $((x_0 \cdot T)\circ e_0)(\overset{t_0}{1}) = (x_0 \cdot T)(0,t_0) = (x_0 \cdot T)(0,1)$

$= 0x_0 + 1 \cdot T(1) = T(1)$

$\Rightarrow (x_0 \cdot T)\circ e_0 = T$

$n=0,\ j=1:\quad ((x_0\cdot T)\circ e^1)(\overset{t_0}{\underset{\uparrow}{1}}) = (x_0\cdot T)(1,0) = x_0$

$n\geq 1,\ j=0:\quad ((x_0\cdot T)\circ e^0)(t_0,\dots,t_n) = (x_0\cdot T)(0,t_0,\dots,t_n)$

$$= T(t_0,\dots,t_n)$$

$$\Rightarrow (x_0\cdot T)\circ e^0 = T$$

$n\geq 1,\ 1\leq j\leq n+1:\quad ((x_0\cdot T)\circ e^j)(t_0,\dots,t_n) = (x_0\cdot T)(t_0,\dots,t_{j-1},0,t_j,\dots,t_n)$

$$= \begin{cases} x_0, & \text{if } t_0=1 \\ t_0x_0+(1-t_0)\,T\Big(\underbrace{\frac{t_1}{1-t_0},\dots,\frac{t_n}{1-t_0}}_{j^{th}\text{ coord.}=0}\Big), & \text{if } 0\leq t_0<1 \end{cases}$$

Also, $\quad x_0\cdot(Te^{j-1})(t_0,\dots,t_n)$

$$= \begin{cases} x_0, & \text{if } t_0=1 \\ t_0x_0+(1-t_0)\,Te^{j-1}\Big(\frac{t_1}{1-t_0},\dots,\frac{t_n}{1-t_0}\Big), & \text{if } 0\leq t_0<1 \end{cases}$$

$$= \begin{cases} x_0, & \text{if } t_0=1 \\ t_0x_0+(1-t_0)\,T\Big(\frac{t_1}{1-t_0},\dots,0,\frac{t_j}{1-t_0},\dots,\frac{t_n}{1-t_0}\Big), & \text{if } 0\leq t_0<1 \end{cases}$$

Thus $\quad (x_0\cdot T)\circ e^j = x_0\cdot(Te^{j-1})$.

Therefore, for $n\geq 1$

$$\underbrace{\partial(x_0\cdot T)}_{\partial\circ(x_0,\cdot)(T)} = \sum_{j=0}^{n+1}(-1)^j(x_0\cdot T)\circ e^j = (x_0\cdot T)\circ e^0 + \sum_{j=1}^{n+1}(-1)^j\underbrace{(x_0\cdot T)\circ e^j}_{x_0\cdot(Te^{j-1})}$$

$$= \underbrace{(x_0\cdot T)\circ e^0}_{=T} - \sum_{i=0}^{n}(-1)^i\underbrace{x_0\cdot(T\circ e^i)}_{(x_0,\cdot)(T\circ e^i)} \qquad (i=j-1)$$

$$= T-(x_0,\cdot)\Big(\underbrace{\sum_{i=0}^{n}(-1)^iT\circ e^i}_{\partial T}\Big)$$

$$= T-(x_0,\cdot)(\partial T).$$

$$\Rightarrow \partial\circ(x_0,\cdot) = id-(x_0,\cdot)\circ\partial,\quad n\geq 1$$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
S_{n+1}(X) \arrow[r, "id"] \arrow[d, "\partial"'] & S_{n+1}(X) \arrow[d, "\partial"] \\
S_n(X) \arrow[r, "id"] \arrow[ur, "{(x_0;\cdot)}"'] \arrow[d, "\partial"'] & S_n(X) \arrow[d, "\partial"] \\
S_{n-1}(X) \arrow[r, "id"] \arrow[ur, "{(x_0,\cdot)}"'] \arrow[d, "\partial"'] & S_{n-1}(X) \arrow[d, "\partial"] \\
\vdots & \vdots
\end{tikzcd}$$

1) $\partial \circ (x_0, \cdot) + (x_0, \cdot) \circ \partial = id$

We show that $Z_n(X) = B_n(X) \quad \forall n \geq 1$

Let $c \in Z_n(X)$. Then $\partial c = 0$.

1) $\Rightarrow \partial \circ (x_0, \cdot)(c) + \underbrace{(x_0, \cdot)\, \partial c}_{=0} = id(c) = c$

$\Rightarrow c = \partial \big(\underbrace{(x_0, \cdot)(c)}_{\in S_{n+1}(X)} \big) \Rightarrow c \in B_n(X).$

$\therefore H_n(X) = Z_n(X) / B_n(X) = 0$

$\forall n \geq 1.$

17. Homotopy invariance of homology

In this section we prove the following result:

Theorem 17.1. Let $f, g : (X, A) \to (Y, B)$ be homotopic functions (i.e., $\exists$ homotopy $F : X \times I \to Y$ s.t. $F_0 = f$, $F_1 = g$ and $F(a, t) \in B \ \forall (a, t) \in A \times I$). Then the induced homomorphisms $f_\#, g_\# : S(X, A) \to S(Y, B)$ are chain homotopic.

Proposition 15.2 and Theorem 17.1 imply:

Theorem 17.2. If $f, g : (X, A) \to (Y, B)$ are homotopic, then

$$f_* = g_* : H_n(X, A) \to H_n(Y, B), \text{ for every } n.$$

$\square$

Theorem 17.3. If $f : X \to Y$ is a homotopy equivalence, then

$$f_* : H_n(X) \to H_n(Y)$$

is an isomorphism, for every n.

proof. Let $g : Y \to X$ be such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. Theorem 17.2 $\Rightarrow$

$$g_* \circ f_* = (g \circ f)_* = (id_X)_* = id : H_n(X) \to H_n(X) \quad \forall n$$

and

$$f_* \circ g_* = (f \circ g)_* = (id_Y)_* = id : H_n(Y) \to H_n(Y) \quad \forall n.$$

Therefore, f_* is an isomorphism with the inverse $(f_*)^{-1} = g_*$, for every n. $\square$

Remark. Theorem 17.2 says that Eilenberg-Steenrod axiom 5 (Homotopy axiom) holds for singular homology.

We now return to the proof of Theorem 17.1. There are different proofs of the result, unfortunately all of them are rather long. We will do a proof based on induction.

<u>proof of Theorem 17.1.</u> We will first show that if $f \simeq g : X \to Y$, then $f\#$ and $g\#$ $S(X) \to S(Y)$ are chain homotopic. The chain homotopy that we will construct has a certain naturality property that then implies the claim in the relative case. Define

$$i_0 : X \to X \times I, \quad i_0(x) = (x, 0) \qquad \forall x \in X$$

and

$$i_1 : X \to X \times I, \quad i_1(x) = (x, 1) \qquad \forall x \in X.$$

Then $id : X \times I \to X \times I$ is a homotopy $i_0 \simeq i_1$.

<u>Claim</u>: To show that $f\#, g\# : S(X) \to S(Y)$ are chain homotopic, it suffices to prove that $(i_0)\#$ and $(i_1)\# : S(X) \to S(X \times I)$ are chain homotopic.

<u>proof of the claim:</u> Let $F : X \times I \to Y$, $F : f \simeq g$. Assume $D : S(X) \to S(X \times I)$ is a chain homotopy from $(i_0)\#$ to $(i_1)\#$:

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+2}(X) \arrow[r, "\partial_{n+2}"] \arrow[d] & S_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d] \arrow[dl, "D_{n+1}"'] & S_n(X) \arrow[r, "\partial_n"] \arrow[d] \arrow[dl, "D_n"'] & S_{n-1}(X) \arrow[r] \arrow[d, "(i_0)\#,\ (i_1)\#"] \arrow[dl, "D_{n-1}"'] & \cdots \\
\cdots \arrow[r] & S_{n+2}(X \times I) \arrow[r, "\partial'_{n+2}"'] & S_{n+1}(X \times I) \arrow[r, "\partial'_{n+1}"'] & S_n(X \times I) \arrow[r, "\partial'_n"'] & S_{n-1}(X \times I) \arrow[r] & \cdots
\end{tikzcd}$$

where

$$\partial'_{n+1} D_n + D_{n-1} \partial_n = (i_1)\# - (i_0)\#.$$

Let

$$F\# : S(X \times I) \to S(Y)$$

be the chain map induced by $F : X \times I \to Y$.

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+2}(X\times I) \arrow[r, "\partial'_{n+2}"] \arrow[d, "F_\#"] & S_{n+1}(X\times I) \arrow[r, "\partial'_{n+1}"] \arrow[d, "F_\#"] & S_n(X\times I) \arrow[r, "\partial'_n"] \arrow[d, "F_\#"] & S_{n-1}(X\times I) \arrow[r] \arrow[d, "F_\#"] & \cdots \\
\cdots \arrow[r] & S_{n+2}(Y) \arrow[r, "\partial''_{n+2}"'] & S_{n+1}(Y) \arrow[r, "\partial''_{n+1}"'] & S_n(Y) \arrow[r, "\partial''_n"'] & S_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Then

$$\begin{aligned}
F_\#(\partial' D + D\partial) &= F_\#((i_1)_\# - (i_0)_\#) \\
&= F_\#(i_1)_\# - F_\#(i_0)_\# \\
&= (\underbrace{F\circ i_1}_{g})_\# - (\underbrace{F\circ i_0}_{f})_\# \\
&= g_\# - f_\#.
\end{aligned}$$

Since $F_\#$ is a chain map, it follows that

$$\partial'' F_\# = F_\# \partial'.$$

Therefore,

$$\begin{aligned}
&\partial''(F_\# D) + (F_\# D)\partial \\
&= F_\# \partial' D + F_\# D \partial \\
&= F_\#(\partial' D + D\partial) = g_\# - f_\#.
\end{aligned}$$

Thus $F_\# \circ D : S(X) \to S(Y)$ is a chain homotopy between $g_\#$ and $f_\#$.

$$\begin{tikzcd}
\cdots \arrow[r] & S_{n+2}(X) \arrow[r, "\partial_{n+2}"] \arrow[d] & S_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d] \arrow[dl, "D_{n+1}"'] & S_n(X) \arrow[r, "\partial_n"] \arrow[d] \arrow[dl, "D_n"'] & S_{n-1}(X) \arrow[r] \arrow[d, "(i_0)_\#,\ (i_1)_\#"] \arrow[dl, "D_{n-1}"'] & \cdots \\
\cdots \arrow[r] & S_{n+2}(X\times I) \arrow[r, "\partial'_{n+2}"] \arrow[d, "F_\#"] & S_{n+1}(X\times I) \arrow[r, "\partial'_{n+1}"] \arrow[d, "F_\#"] & S_n(X\times I) \arrow[r, "\partial'_n"] \arrow[d, "F_\#"] & S_{n-1}(X\times I) \arrow[r] \arrow[d, "F_\#"] & \cdots \\
\cdots \arrow[r] & S_{n+2}(Y) \arrow[r, "\partial''_{n+2}"'] & S_{n+1}(Y) \arrow[r, "\partial''_{n+1}"'] & S_n(Y) \arrow[r, "\partial''_n"'] & S_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

This proves the Claim 1.

We next prove that $(i_0)_\#$ and $(i_1)_\#$ are chain homotopic. In fact, we prove a stronger result:

Claim 2. For every topological space X there is a chain homotopy $D_X : S(X) \to S(X \times I)$ between $(i_0)_\#$ and $(i_1)_\#$ satisfying the following naturality condition: If $h : X \to X'$ is an arbitrary continuous function, then the diagram

$$\begin{tikzcd} S_n(X) \arrow[r, "D_X"] \arrow[d, "h_\#"'] & S_{n+1}(X \times I) \arrow[d, "(h \times id)_\#"] \\ S_n(X') \arrow[r, "D_{X'}"'] & S_{n+1}(X' \times I) \end{tikzcd} \qquad (*)$$

commutes for every n.

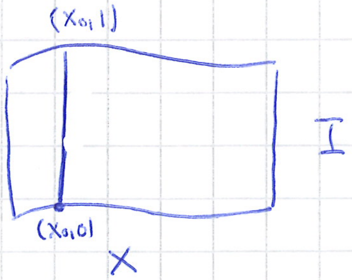
Proof of Claim 2: The proof is done by induction on n.

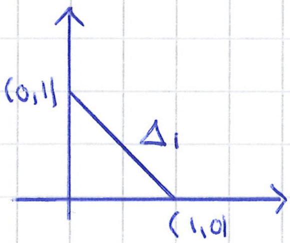
$n=0$: Let $T : \Delta_0 \to X$ be a singular 0-simplex of X. Define a singular 1-simplex $D_X T$ of $X \times I$ by

$$D_X T : \Delta_1 \to X \times I, \quad (D_X T)(t_0, t_1) = (T(\Delta_0), t_1)$$

Here $t_0 + t_1 = 1$, $t_0, t_1 \geq 0$

$\Delta_0 = \{1\} \mapsto T(\Delta_0) = x_0$





Find the boundary of $D \times T$:

$$(D\times T)\circ e^0 : \underset{\{1\}}{\overset{}{\Delta_0}} \to X\times I$$

$$\Delta_0 = \{1\}$$

$$(D\times T)e^0(1) = (D\times T)(0,1) = (T(\underbrace{\Delta_0}_{\{1\}}),1) = (i_1\circ T)(1)$$

$$(D\times T)e^1(1) = (D\times T)(1,0) = (T(\Delta_0),0) = (i_0\circ T)(1).$$

$$\partial'(D\times T) = (D\times T)\circ e_0 - (D\times T)\circ e_1 = i_1\circ T - i_0\circ T$$

$$= (i_1)_\#(T) - (i_0)_\#(T) = ((i_1)_\# - (i_0)_\#)(T).$$

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
S_0(X) \arrow[rr, "D_X"] \arrow[d, "\partial"'] \arrow[drr, shift left, "(i_0)_\#"] \arrow[drr, shift right, "(i_1)_\#"'] & & S_1(X\times I) \arrow[d, "\partial'"] \\
S_{-1}(X) \arrow[rr, "D_X"'] \arrow[d, equal] & & S_0(X\times I) \\
0 & &
\end{tikzcd}$$

$$\partial' D_X + \underbrace{D_X\partial}_{0} = \partial' D_X$$

$$= (i_1)_\# - (i_0)_\#$$

o.k.

Also Condition $(*)$ holds:

Let $h: X \to X'$ be a continuous function. Then

$$((h\times id)_\# D_X)(T) = (h\times id)_\#(D\times T) = (h\times id)\circ D\times T : \Delta_1 \to X'\times I$$

$$\Rightarrow ((h\times id)_\# D\times T)(t_0,t_1) = (h\times id)(D\times T(t_0,t_1)) = (h\times id)(T(\Delta_0),t_1)$$
$$= (h\,T(\Delta_0), t_1)$$

also $(D_{X'} h_\#)(T) = D_{X'}(h\circ T)$

$$\Rightarrow (D_{X'} h_\#)(T)(t_0,t_1) = D_{X'}(h\circ T)(t_0,t_1) = (h\circ T(\Delta_0), t_1)$$

Thus $D_{X'}\circ h_\# = (h\times id)_\#\circ D_X$, i.e. $(*)$ commutes for $n=0$.

$\therefore$ The claim for $n=0$.