Introduction to algebraic topology (10 cr)

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Lectures: Tue 10:15 – 12:00 in B322
Thu 10:15 – 12:00 in B322

Exercises: Wed 10:15 – 12:00 in B322

- The first meeting will be on Wed, Sept 9.

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Topics:

- definition of homotopy
- contractibility
- path connectedness
- simplexes
- the fundamental group
- singular homology
- long exact sequences
- excision

Notation:

$\mathbb{Z}$ = integers
$\mathbb{Q}$ = rational numbers
$\mathbb{R}$ = real numbers
$\mathbb{C}$ = complex numbers
$I = [0,1]$, the closed unit interval
$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \text{ for all } i\}$
$I^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in I \text{ for all } i\}$

$x \in \mathbb{R}$: the absolute value of x is $|x|$

$x \in \mathbb{R}^n$: the norm of x is $\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$,

where $x = (x_1, \dots, x_n)$

$S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$, the unit sphere, also called the n-sphere of radius 1 and center the origin

$S^0 = \{ x \in \mathbb{R} \mid |x| = 1 \} = \{-1, 1\}$

$D^n = \{ x \in \mathbb{R}^n \mid \|x\| < 1 \}$, the closed n-disk, also called the closed n-ball

1. Categories and Functors

Definition 1.1. A category $\overline{C}$ consists of three ingredients:

1) a class of objects, $obj(\overline{C})$

2) a class of morphisms, $Hom(\overline{C})$

3) composition of morphisms

For every ordered pair $A, B \in obj(\overline{C})$, there is a set of morphisms $Hom(A, B)$.

For $A, B, C \in obj(\overline{C})$, there is composition of morphisms

$$Hom(A,B) \times Hom(B,C) \to Hom(A,C)$$
$$(f, g) \mapsto g \circ f$$

The following axioms are satisfied:

i) the family of $Hom(A,B)$'s is pairwise disjoint

ii) composition is associative

$$(h \circ (g \circ f) = (h \circ g) \circ f)$$

iii) for every object A, there exits a morphism $1_A : A \to A$ called the identity morphism for A, s.t. $1_A \circ f = f \quad \forall f \in Hom(B, A), \forall B \in obj(\overline{C})$

and $g \circ 1_A = g \quad \forall\, g \in \mathrm{Hom}(A,C),\ \forall\, C \in \mathrm{obj}(\bar{C})$.

<u>Example</u> $\bar{C} = \mathrm{Sets}$ (= the category of all sets)

$\mathrm{obj}(\bar{C})$ = all sets

$\mathrm{Hom}(A,B)$ = the family of all functions $f: A \to B$

composition = the usual composition of functions

<u>Example</u> $\bar{C} = \mathrm{Top}$

$\mathrm{obj}(\bar{C})$ = all topological spaces

$\mathrm{Hom}(A,B)$ = the family of all continuous functions $A \to B$

composition = the usual composition of functions

<u>Definition 1.2.</u> Let $\bar{A}$ and $\bar{C}$ be categories. Assume $\mathrm{obj}(\bar{C}) \subset \mathrm{obj}(\bar{A})$.
For $A, B \in \mathrm{obj}(\bar{C})$, denote the sets of morphisms corresponding to $\bar{A}$ and $\bar{C}$ by

$$\mathrm{Hom}_{\bar{A}}(A,B) \quad \text{and} \quad \mathrm{Hom}_{\bar{C}}(A,B),$$

respectively. We call $\bar{C}$ a subcategory of $\bar{A}$, if

$$\mathrm{Hom}_{\bar{C}}(A,B) \subset \mathrm{Hom}_{\bar{A}}(A,B),$$

for all $A, B \in \mathrm{obj}(\bar{C})$ and if the composition in $\bar{C}$ is the same as the composition in $\bar{A}$.

Example Subcategories of Top.

We obtain subcategories by restriction. For example, the objects can be Hausdorff spaces or normal spaces or compact spaces... If we choose the objects to be smooth manifolds, we may choose the morphisms to be smooth maps.

Example $\bar{C}$ = Groups

obj($\bar{C}$) = all groups

Hom(A,B) = the family of all homomorphisms $A \to B$

composition = the usual composition

Example $\bar{C}$ = Ab

obj($\bar{C}$) = all Abelian groups

Hom(A,B) = the family of all homomorphisms $A \to B$

composition = the usual composition

Then Ab is a subcategory of Groups.

Example $\bar{C}$ = Rings

obj($\bar{C}$) = all rings

Hom(A,B) = all ring homomorphisms $A \to B$ that preserve identity elements

composition = the usual composition

<u>Example</u> $\bar{C} = Top^2$

$obj(\bar{C})$ = all ordered pairs (X, A), where X is a topological space and A is a subspace of X

$Hom((X,A),(Y,B))$: a morphism

$$f:(X,A) \to (Y,B)$$

is a continuous map $f: X \to Y$ with $f(A) \subset B$.

<u>Example</u> $\bar{C} = Top_*$

$obj(\bar{C})$ = all ordered pairs (X, x_0), where X is a topological space and $x_0 \in X$.

A morphism $f:(X, x_0) \to (Y, y_0)$ is a continuous map $f: X \to Y$ with $f(x_0) = y_0$.

Here, x_0 = the <u>basepoint</u> of (X, x_0)

Morphisms are called <u>pointed maps</u> or basepoint preserving maps. Objects are called <u>pointed spaces</u>. Top_* is called the category of <u>pointed spaces</u>, Top_* is a subcategory of Top^2.

Definition 1.3. Let $\bar{C}$ be a category. Let $\sim$ be an equivalence relation on

$$\bigcup_{(A,B)} Hom(A,B).$$

We call $\sim$ a congruence on $\bar{C}$, if it satisfies the following conditions:

i/ if $f \in Hom(A,B)$ and $f \sim f'$, then $f' \in Hom(A,B)$

ii/ if $f \sim f'$, $g \sim g'$ and the composite $g \circ f$ exists, then $g \circ f \sim g' \circ f'$.

Theorem 1.4. Let $\bar{C}$ be a category and let $\sim$ be a congruence on $\bar{C}$. Let $[f]$ denote the equivalence class of a morphism f. Define $\bar{C}'$ by

$obj(\bar{C}') = obj(\bar{C})$,

$Hom_{\bar{C}'}(A,B) = \{[f] \mid f \in Hom_{\bar{C}}(A,B)\}$,

$[g] \circ [f] = [g \circ f]$.

Then $\bar{C}'$ is a category. $\square$

The category $\bar{C}'$ is called a <u>quotient category</u> of $\bar{C}$.

For us, the most important quotient category will be the <u>homotopy category</u>.

Definition 1.5. Let $\bar{A}$ and $\bar{C}$ be categories. Let $T: \bar{A} \to \bar{C}$ satisfy the following:

1) if $A \in obj(\bar{A})$, then $TA \in obj(\bar{C})$,

2) if $f: A \to A'$ is a morphism in $\bar{A}$, then
$$Tf: TA \to TA' \quad \text{is a morphism in } \bar{C}$$

3) if f, g are morphisms in $\bar{A}$ and $g \circ f$ is defined, then
$$T(g \circ f) = (Tg) \circ (Tf),$$

4) $T(1_A) = 1_{TA}$ for every $A \in obj(\bar{A})$.

We say that T is a <u>(covariant) functor</u> from $\bar{A}$ to $\bar{C}$.

<u>Example</u> Let $\bar{A}$ and $\bar{C}$ be categories and let $T: \bar{A} \to \bar{C}$ be a functor. We call T a <u>forgetful functor</u>, if it "forgets" some of the structure or properties of $\bar{A}$. The functor $T: Top \to Sets$ that assigns to each topological space its underlying set and to each continuous function itself is an example of a forgetful functor.

<u>Example</u> Let $\bar{C}$ be a category. The <u>identity functor</u> $J: \bar{C} \to \bar{C}$ is defined by $JA = A$ for every $A \in obj(\bar{C})$ and $Jf = f$ for every morphism f.

Example Let Y be a topological space. Then there is a functor

$$T_Y : Top \longrightarrow Top,$$

where $T_Y(X) = X \times Y$, for a topological space X,

and, for a continuous function $f : X \to X'$,

$$T_Y(f) : X \times Y \longrightarrow X' \times Y$$

is defined by $(x, y) \longmapsto (f(x), y)$.

Example Let $\overline{C}$ be a category. Let $A \in obj(\overline{C})$. Define a functor

$$Hom(A, \ ) : \overline{C} \longrightarrow Sets$$

as follows: assign the set $Hom(A, B)$ to each $B \in obj(\overline{C})$ and assign the induced map

$$Hom(A, f) : Hom(A, B) \longrightarrow Hom(A, B'),$$
$$g \longmapsto f \circ g$$

to every morphism $f : B \to B'$.

A functor is called <u>contravariant</u>, if it reverses the direction of arrows.

The functor in the example above is called a <u>covariant Hom functor</u>.

<u>Definition 1.6.</u> Let $\bar{A}$ and $\bar{C}$ be categories. Let $S: \bar{A} \to \bar{C}$ satisfy the following:

1) if $A \in obj(\bar{A})$, then $SA \in obj(\bar{C})$,

2) if $f: A \to A'$ is a morphism in $\bar{A}$, then

$$Sf: SA' \to SA \text{ is a morphism in } \bar{C},$$

3) if f, g are morphisms in $\bar{A}$ and $g \circ f$ is defined, then
$$S(g \circ f) = S(f) \circ S(g),$$

4) $S(1_A) = 1_{SA}$ for every $A \in obj(\bar{A})$.

We say that S is a <u>contravariant functor</u> from $\bar{A}$ to $\bar{C}$.

<u>Example</u> Let $\bar{C}$ be a category. Let $B \in obj(\bar{C})$. Define a functor

$$Hom(\ , B): \bar{C} \to Sets$$

as follows: assign the set $Hom(A, B)$ to each $A \in obj(\bar{C})$ and assign the induced map

$$Hom(g, B): Hom(A', B) \to Hom(A, B),$$
$$h \mapsto h \circ g$$

to every morphism $g: A \to A'$.

The functor $Hom(\ , B)$ is called a <u>contravariant Hom functor.</u>

Definition 1.7. Let $\bar{C}$ be a category and let $A, B \in \bar{C}$. Let $f: A \to B$ be a morphism. If there is a morphism $g: B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$, we say that f is an <u>equivalence in $\bar{C}$</u>.

Theorem 1.8. Let $\bar{A}$ and $\bar{C}$ be categories and let $T: \bar{A} \to \bar{C}$ be either a covariant or contravariant functor. Let f be an equivalence in $\bar{A}$. Then Tf is an equivalence in $\bar{C}$.

proof. Assume T is covariant. Let $f: A \to B$ be a morphism that is an equivalence in $\bar{A}$. Then there is a morphism $g: B \to A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$. Since T is covariant, it follows that

$$T(g) \circ T(f) = T(g \circ f) = T(1_A) = 1_{TA}, \quad \text{and}$$

$$T(f) \circ T(g) = T(f \circ g) = T(1_B) = 1_{TB}.$$

Thus Tf is an equivalence in $\bar{C}$. The case where T is contravariant is similar. $\square$

2. Homotopy

Definition 2.1. Let X and Y be topological spaces and let f_0 and f_1 be continuous maps from X to Y. We say that f_0 is <u>homotopic</u> to f_1 (denoted by $f_0 \simeq f_1$), if there is a continuous map

$$F : X \times I \to Y \quad \text{such that}$$

$$F(x,0) = f_0(x) \quad \text{and} \quad F(x,1) = f_1(x) \quad \text{for all } x \in X.$$

The map F is called ~~to~~ <u>homotopy.</u>

Notation: We often write $F : f_0 \simeq f_1$.

Let $t \in I$. Define $f_t : X \to Y$ by $f_t(x) = F(x,t)$. Thus the homotopy F gives a <u>one-parameter family</u> of continuous maps deforming f_0 into f_1.

Recall the following <u>gluing lemmas:</u>

Lemma 2.1. (Gluing lemma 1) Let X be a topological space and let X_i, $1 \leq i \leq n$, be finitely many closed subsets of X such that $X = \bigcup_{i=1}^{n} X_i$. Let Y be a topological space. Assume there are continuous maps $f_i : X_i \to Y$, $1 \leq i \leq n$ such that

$$f_i | X_i \cap X_j = f_j | X_i \cap X_j, \quad \text{for all } i,j.$$

Then there exists a unique continuous map $f : X \to Y$ with $f | X_i = f_i$ for all i.

Lemma 2.1': (Glueing lemma 2) Let X be a topological space and let X_i be (possibly infinitely many) open subsets of X such that $X = \bigcup X_i$. Let Y be a topological space. Assume there are continuous maps $f_i : X_i \to Y$ such that

$$f_i|X_i \cap X_j = f_j|X_i \cap X_j, \quad \text{for all } i, j.$$

Then there exists a unique continuous map $f: X \to Y$ with $f|X_i = f_i$ for all i.

For proof of Lemma 2.1 and 2.1', see Rotman, Lemma 1.1.

Theorem 2.2. Homotopy is an equivalence relation on the set of all continuous maps $X \to Y$.

Proof. 1) Reflexivity: Let $f: X \to Y$ be a continuous map. Then

$$F: X \times I \to Y, \ (x,t) \mapsto f(x), \ \text{for all } x \in X \text{ and for all } t \in I,$$

is a homotopy, $F: f \simeq f$.

2) Symmetry: Let $f, g : X \to Y$ and assume $f \simeq g$. Then there is a continuous map

$$F: X \times I \to Y, \ F(x,0) = f(x), \ F(x,1) = g(x) \ \text{for all } x \in X.$$

Let

$$G: X \times I \to Y, \ G(x,t) = F(x, 1-t).$$

Then $G: g \simeq f$.

<u>Transitivity:</u> Let $f, g, h : X \to Y$ and assume $F : f \simeq g$ and $G : g \simeq h$. Let

$$H : X \times I \to Y, \quad (x,t) \mapsto \begin{cases} F(x, 2t), & \text{if } 0 \le t \le \frac{1}{2} \\ G(x, 2t-1), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Now, for $t = \frac{1}{2}$,

$$F(x, 2 \cdot \tfrac{1}{2}) = F(x, 1) = g(x) \quad \text{for every } x \in X$$

and

$$G(x, 2 \cdot \tfrac{1}{2} - 1) = G(x, 0) = g(x) \quad \text{for every } x \in X.$$

Thus it follows from the gluing lemma (2.1), that H is continuous. Therefore, $H : f \simeq h$.

$\square$

<u>Definition 2.3.</u> The <u>homotopy class</u> of a continuous map $f : X \to Y$ is the equivalence class

$$[f] = \{ \text{continuous } g : X \to Y : g \simeq f \}.$$

The family of all such homotopy classes is denoted by $[X, Y]$.

<u>Theorem 2.4.</u> Let $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$ be continuous. Assume $f_0 \simeq f_1$ and $g_0 \simeq g_1$. Then $g_0 \circ f_0 \simeq g_1 \circ f_1$, i.e., $[g_0 \circ f_0] = [g_1 \circ f_1]$.

<u>proof.</u> Let $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$ be homotopies. Let

$$H : X \times I \to Z, \quad (x,t) \mapsto G(f_0(x), t).$$

Then H is continuous, $H(x,0) = G(f_0(x),0) = g_0(f_0(x))$ and $H(x,1) = G(f_0(x),1) = g_1(f_0(x))$, for all $x \in X$. Then $H: g_0 \circ f_0 \simeq g_1 \circ f_0$. Let then

$$K: X \times I \to Z,\ (x,t) \mapsto (g_1 \circ F)(x,t).$$

Thus $K: g_1 \circ f_0 \simeq g_1 \circ f_1$. Since homotopy is a transitive relation, it follows that $g_0 \circ f_0 \simeq g_1 \circ f_1$.

$\square$

The following corollary follows immediately from the definition of congruence and from Theorems 2.2 and 2.4:

<u>Corollary 2.5.</u> Homotopy is a congruence on the category Top. $\square$

According to Theorem 1.4, there is a quotient category whose objects are topological spaces and whose morphism sets are $\mathrm{Hom}(X,Y) = [X,Y]$. The composition is given by $[g] \circ [f] = [g \circ f]$. This category is called the <u>homotopy category</u> and it is denoted by $h\mathrm{Top}$.

<u>Definition 2.6.</u> A continuous map $f: X \to Y$ is called a <u>homotopy equivalence</u>, if there exists a continuous map $g: Y \to X$ with $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_Y$. Topological spaces X and Y have the same <u>homotopy type</u>, if there is a homotopy equivalence $f: X \to Y$.

Clearly, $f: X \to Y$ is a homotopy equivalence if and only if $[f] \in [X,Y]$ is an equivalence in $h\mathrm{Top}$.

Definition 2.7. Let X and Y be topological spaces and let $y_0 \in Y$. The map $c: X \to Y$, $x \mapsto y_0$, is called the constant map at y_0. A continuous map $f: X \to Y$ is called <u>null homotopic</u>, if there exists a constant map $c: X \to Y$ with $f \simeq c$.

Theorem 2.8. Let Y be a topological space and let $f: S^n \to Y$ be a continuous map. The following conditions are equivalent:

1) f is nullhomotopic,
2) f can be extended to a continuous map $D^{n+1} \to Y$,
3) if $x_0 \in S^n$ and $k: S^n \to Y$ is the constant map at $f(x_0)$, then there is a homotopy $F: f \simeq k$ with $F(x_0, t) = f(x_0)$ for all $t \in I$.

<u>proof.</u> 1) $\Rightarrow$ 2): Let $c: S^n \to Y$, $x \mapsto y_0$, where $y_0 \in Y$. Assume $F: f \simeq c$. Let

$$g: D^{n+1} \to Y, \quad x \mapsto \begin{cases} y_0, & \text{if } 0 \le \|x\| \le \frac{1}{2} \\ F(x/\|x\|, 2 - 2\|x\|), & \text{if } \frac{1}{2} \le \|x\| \le 1. \end{cases}$$

The map g is well defined since for $\|x\| = \frac{1}{2}$,

$$F(x/\|x\|, 2 - 2\|x\|) = F(\underset{\in S^n}{2x}, 1) = y_0.$$

By the gluing lemma, g is continuous. If $x \in S^n$, then $\|x\| = 1$ and

$$g(x) = F(x, 2 - 2\cdot 1) = F(x, 0) = f(x).$$

Thus g is an extension of f.

2) ⇒ 3) : Assume that $g: D^{n+1} \to Y$ extends f. Let

$$F: S^n \times I \to Y, \ (x,t) \mapsto g(\underbrace{(1-t)x + t x_0}_{\in D^{n+1}}).$$

Clearly, F is continuous. For all $x \in S^n$,

$$F(x,0) = g(x) = f(x) \quad \text{and} \quad F(x,1) = g(x_0) = f(x_0).$$

Thus $F: f \simeq k$, where $k: S^n \to Y$, $x \mapsto f(x_0)$.
For all $t \in I$,

$$F(x_0,t) = g((1-t)x_0 + t x_0) = g(x_0) = f(x_0).$$

3) ⇒ 1) obvious □

3. Convexity, contractibility and cones

Definition 3.1. A subset X of $\mathbb{R}^m$ is called convex, if $tx + (1-t)y \in X$ for all $x, y \in X$ and for all $t \in I$.

Definition 3.2. A topological space X is called contractible if the identity map $1_X: X \to X$ is nullhomotopic.

Theorem 3.3. Every convex set is contractible.

proof. Let X be a convex set and let $x_0 \in X$. Define $c: X \to X$ by $c(x) = x_0$ for all $x \in X$.
Define $F: X \times I \to X$ by $F(x,t) = t x_0 + (1-t)x$.
Then $F: 1_X \simeq c$. □