8. The Functor π_1

Top_* : the category of pointed spaces (X, x_0)
morphisms: continuous functions $f : X \to Y$, $f(x_0) = y_0$; where x_0, y_0 are the basepoints of X and Y, respectively.

S^1 : basepoint 1
I : basepoint 0

Theorem 8.1. $\pi_1 : Top_* \to Groups$ is a functor. If $h, k : (X, x_0) \to (Y, y_0)$ and $h \simeq k$ rel $\{x_0\}$, then $\pi_1(h) = \pi_1(k)$.

proof. Define

$\pi_1(h) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $[f] \mapsto [h \circ f]$.

Since $h \circ f : I \to Y$ is continuous and $(h \circ f)(0) = h(f(0)) = h(x_0) = y_0 = h(f(1)) = (h \circ f)(1)$, it follows that $[h \circ f] \in \pi_1(Y, y_0)$.

If $f \simeq f'$ rel $\dot{I}$, then Exercise 4.1 $\Rightarrow$ $h \circ f \simeq h \circ f'$ rel $\dot{I}$. Thus $\pi_1(h)$ is well defined.

Let f, g be closed paths in X at x_0, then

$$h \circ (f * g) = (h \circ f) * (h \circ g).$$

$\Rightarrow$ $\pi_1(h)$ is a homomorphism.

Let $id_X : X \to X$ be the identity. Then $\pi_1(id_X) : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity. Also, for $h : (X, x_0) \to (Y, y_0)$ and $l : (Y, y_0) \to (Z, z_0)$, $\pi_1(l \circ h) = \pi_1(l) \circ \pi_1(h)$.

$\therefore$ π_1 is a functor

Let f be a closed path in X at x_0. Let $h \simeq k$ rel $\{x_0\}$.
Then Ex. 4.1 $\Rightarrow$ $h \circ f \simeq k \circ f$ rel $\dot{I}$.
$\Rightarrow [h \circ f] = [k \circ f]$.
$\therefore \pi_1(h) = \pi_1(k)$.
□

Notation: Write h_* for $\pi_1(h)$, h_* is the map <u>induced</u> by h.

<u>Pointed homotopy category $hTop_*$:</u>

Congruence of relative homotopy: $f_0 \simeq f_1$ rel $\{x_0\}$ for $f_0, f_1 : (X, x_0) \to (Y, y_0)$.
<u>Thus: Objects</u> of $hTop_*$: pointed spaces (X, x_0)
<u>Morphisms</u> $(X, x_0) \to (Y, y_0)$: relative homotopy classes $[f]$, where $f: (X, x_0) \to (Y, y_0)$ is a pointed map.
<u>Composition:</u> $[h \circ f] = [h][f]$, for h, f s.t. composing in Top_* is possible.

<u>Theorem 8.2.</u> Let $x_0 \in X$ and let X_0 be the path component of X containing x_0.
Then
$$\pi_1(X, x_0) \cong \pi_1(X_0, x_0).$$

<u>proof.</u> Let $i : (X_0, x_0) \to (X, x_0)$ be the inclusion. Then i induces the homomorphism $i_* : \pi_1(X_0, x_0) \to \pi_1(X, x_0)$. Assume $[f] \in \ker i_*$. Then $i \circ f \simeq c$ rel $\dot{I}$, where $c: I \to X$, $t \mapsto x_0$, is the constant path at x_0.

$i \circ f$ is nullhomotopic in X_0: Let $F: i \circ f \simeq c$ rel $\dot{I}$.
Then $F(0,0) = (i \circ f)(0) = f(0) = x_0 \in X_0$.
$I \times I$ path conn. $\Rightarrow F(I \times I)$ path conn.
$\Big\} \Rightarrow F(I \times I) \subset X_0$.

$\therefore F$ is injective

i_* is surjective: Let $f: I \to X$ be a closed path at x_0. Then $f(I) \subset X_0$. Let $f': I \to X_0$, $f'(t) = f(t)$ $\forall t \in I$. Then $i \circ f' = f$.
$\Rightarrow$ $i_*[f'] = [i \circ f'] = [f]$.

$\therefore$ i_* is surjective

$\square$

Theorem 8.3. Let X be pathconnected and let $x_0, x_1 \in X$. Then

$$\pi_1(X, x_0) \cong \pi(X, x_1).$$

proof. Since X is pathconnected, there is a path γ in X from x_0 to x_1. Let

$$\varphi: \pi_1(X, x_0) \to \pi_1(X, x_1), \quad [f] \mapsto [\gamma^{-1}][f][\gamma],$$

(multiplication in the groupoid of X).

Theorem 7.9 $\Rightarrow$ φ is an isomorphism, the inverse of φ is given by $[g] \mapsto [\gamma][g][\gamma^{-1}]$. $\square$

Notation: Let H, K, L be sets, $\alpha: L \to H$ and $\beta: L \to K$ be functions. Then there are projections $p: H \times K \to H$, $(h,k) \mapsto h$, and $q: H \times K \to K$, $(h,k) \mapsto k$. There also is a function $(\alpha, \beta): L \to H \times K$, $x \mapsto (\alpha(x), \beta(x))$.

$$\begin{tikzcd}
 & H \times K \arrow[dl, "p"'] \arrow[dr, "q"] & \\
H & & K \\
 & L \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \arrow[uu, "{(\alpha,\beta)}"] &
\end{tikzcd}$$

H, K, L groups
α, β homomorphisms
$\Rightarrow$ (α, β) homomorphism

Theorem 8.4. Let (X, x_0) and (Y, y_0) be pointed spaces. Then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

proof. Let $p : (X \times Y, (x_0, y_0)) \to (X, x_0)$ and $q : (X \times Y, (x_0, y_0)) \to (Y, y_0)$ be projections. Then $(p, q) : (X \times Y, (x_0, y_0)) \to (X, x_0) \times (Y, y_0)$ induces

$$\begin{aligned}(p_*, q_*) : \pi_1(X \times Y, (x_0, y_0)) &\to \pi_1(X, x_0) \times \pi_1(Y, y_0),\\ [\phi] &\mapsto (p_*[\phi], q_*[\phi]) = ([p \circ \phi], [q \circ \phi]),\end{aligned}$$

where ϕ is a closed path at (x_0, y_0). Then (p_*, q_*) is a homomorphism. We show that it is an isomorphism by finding its inverse.

Let $g : I \to X$ be a closed path in X at x_0 and let $h : I \to Y$ be a closed path in Y at y_0. Let

$$\begin{aligned}\theta : \pi_1(X, x_0) \times \pi_1(Y, y_0) &\longrightarrow \pi_1(X \times Y, (x_0, y_0)),\\ ([g], [h]) &\longmapsto [(g, h)],\end{aligned}$$

where $(g, h) : I \to X \times Y$, $t \mapsto (g(t), h(t))$. Exercise 4.5 $\Rightarrow$ θ is well defined. Then

$$[\phi] \xmapsto{(p_*, q_*)} ([p \circ \phi], [q \circ \phi]) \xmapsto{\theta} [\underbrace{(p \circ \phi, q \circ \phi)}_{\phi}] = [\phi]$$

and

$$([g], [h]) \xmapsto{\theta} [(g, h)] \xmapsto{(p_*, q_*)} ([\underbrace{p \circ (g, h)}_{g}], [\underbrace{q \circ (g, h)}_{h}]) = ([g], [h]).$$

Thus θ is the inverse of (p_*, q_*). $\square$

Lemma 8.5. Let $\varphi_0, \varphi_1 : X \to Y$ be continuous. Assume there is a (free) homotopy $F : \varphi_0 \simeq \varphi_1$. Let $x_0 \in X$ and let λ be the path $F(x_0, \ )$ in Y from $\varphi_0(x_0)$ to $\varphi_1(x_0)$. Then there is a commutative diagram

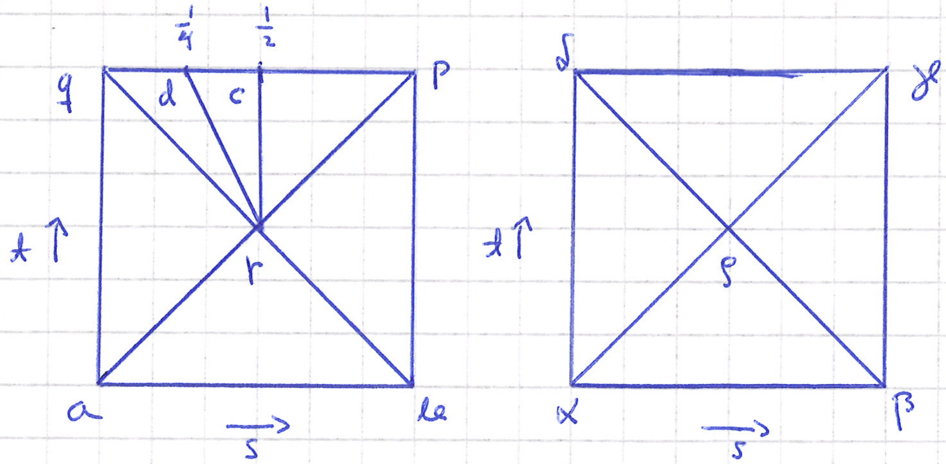
$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\varphi_{1*}"] \arrow[dr, "\varphi_{0*}"'] & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\psi"] \\
& \pi_1(Y, \varphi_0(x_0))
\end{tikzcd}$$

where ψ is the isomorphism $[g] \mapsto [\lambda * g * \lambda^{-1}]$.

proof. Let $f : I \to X$ be a closed path at x_0.
Let

$$G : I \times I \to Y, \quad (s,t) \mapsto F(f(s), t).$$

Then $G : \varphi_0 \circ f \simeq \varphi_1 \circ f$. Consider the following triangulations of $I \times I$:



Define a continuous map $H : I \times I \to I \times I$.
First, define H on each triangle, then use the Gluing lemma.
On every triangle (= 2-simplex), H is an affine map.
$\Rightarrow$ it suffices to evaluate it on each vertex. Then the maps defined on overlaps agree automatically.

Define: $H(a) = H(q) = \alpha$
$H(b) = H(p) = \beta$
$H(c) = \gamma$
$H(d) = \delta$
$H(r) = \rho$.

Then: $[a, q]$ collapses to α
$[b, p]$ collapses to β
$[q, d]$ goes to $[\alpha, \delta]$
$[d, c]$ goes to $[\delta, \gamma]$
$[c, p]$ goes to $[\gamma, \beta]$

Define

$$J = G \circ H : I \times I \longrightarrow Y,$$
$$(s, t) \longmapsto G(H(s,t)).$$

$t=0$: $J(s,0) = G(H(s,0)) = G(s,0) = F(f(s),0) = (\varphi_0 \circ f)(s)$

$t=1$: $J(s,1) = G(H(s,1)) = (\lambda * (\varphi_1 \circ f) * \lambda^{-1})(s)$

$s=0$: $J(0,t) = G(H(0,t)) = G(0,0) = F(f(0),0) = \varphi_0(f(0)) = \varphi_0(x_0)$

$s=1$: $J(1,t) = G(H(1,t)) = G(1,0) = F(f(1),0) = \varphi_0(f(1)) = \varphi_0(x_0)$

Thus $J: \varphi_0 \circ f \simeq \lambda * (\varphi_1 \circ f) * \lambda^{-1}$ rel $\dot{I}$.

Now, $\varphi_{0*}[f] = [\varphi_0 \circ f] = [\lambda * (\varphi_1 \circ f) * \lambda^{-1}]$.

Also, $(\psi \circ \varphi_{1*})[f] = \psi(\varphi_{1*}[f]) = \psi([\varphi_1 \circ f]) = [\lambda * (\varphi_1 \circ f) \circ \lambda^{-1}]$.

It follows that $\varphi_{0*} = \psi \circ \varphi_{1*}$. $\square$

__Consequence:__ Freely homotopic maps φ_0 and φ_1 may not induce the same homomorphism between fundamental groups, they differ by the isomorphism ψ.

__Corollary 8.6.__ Assume $\varphi_0, \varphi_1 : (X, x_0) \to (Y, y_0)$ are freely homotopic. Then:

1) There is $[\lambda] \in \pi_1(Y, y_0)$ s.t. $\varphi_{0*}[f] = [\lambda]\varphi_{1*}[f][\lambda]^{-1}$ for all $[f] \in \pi_1(X, x_0)$. ($\varphi_{0*}, \varphi_{1*}$ are conjugate)

2) If $\pi_1(Y, y_0)$ is abelian, then $\varphi_{0*} = \varphi_{1*}$.

__proof.__ Part 1 follows immediately from Lemma 8.5: $\varphi_0(x_0) = y_0 = \varphi_1(x_0)$, λ is a closed path at y_0.
$\Rightarrow [\lambda] \in \pi_1(Y, y_0)$, and

$$[\lambda * (\varphi_1 \circ f) * \lambda^{-1}] = [\lambda][\varphi_1 \circ f][\lambda^{-1}] = [\lambda]\varphi_{1*}[f][\lambda]^{-1}.$$

2) $\pi_1(Y, y_0)$ abelian $\Rightarrow \varphi_{0*}[f] = [\lambda]\varphi_{1*}[f][\lambda]^{-1}$
$= [\lambda][\lambda]^{-1}\varphi_{1*}[f] = \varphi_{1*}[f]$. $\square$

__Theorem 8.7.__ Let $\beta : X \to Y$ be a homotopy equivalence. Then $\beta_* : \pi_1(X, x_0) \to \pi_1(Y, \beta(x_0))$ is an isomorphism for every $x_0 \in X$.

__proof.__ Since β is a homotopy equivalence, there is a continuous map $\alpha : Y \to X$ with $\alpha \circ \beta \simeq 1_X$ and $\beta \circ \alpha \simeq 1_Y$. Consider the following diagram:

$$\begin{tikzcd}
& \pi_1(Y, \beta(x_0)) \arrow[dr, "\alpha_*"] & \\
\pi_1(X, x_0) \arrow[ur, "\beta_*"] \arrow[rr, "(\alpha\circ\beta)_*"] \arrow[dr, "1"'] & & \pi_1(X, \alpha\beta(x_0)) \arrow[dl, "\Psi"] \\
& \pi_1(X, x_0) &
\end{tikzcd}$$

Lemma 8.5 $\Rightarrow$ the lower triangle commutes

Ψ isomorphism $\Rightarrow (\alpha \circ \beta)_*$ is an isomorphism.

Top triangle: π_1 is a functor $\Rightarrow (\alpha \circ \beta)_* = \alpha_* \circ \beta_*$
$\Rightarrow$ the top triangle commutes.

$(\alpha \circ \beta)_*$ bijection $\Rightarrow \begin{cases} \alpha_* \text{ surjection} \\ \beta_* \text{ injection} \end{cases}$

$\beta \circ \alpha \simeq 1_Y \Rightarrow$ similar diagram that shows α_* is injective, β_* is surjective.

Thus: β_* is an isomorphism. □

Corollary 8.8. Let X and Y be path connected and assume they have the same homotopy type. Then, for every $x_0 \in X$ and for every $y_0 \in Y$,

$$\pi_1(X, x_0) \cong \pi_1(Y, y_0).$$

proof. Let $\beta: X \to Y$ be a homotopy equivalence. Theorem 8.7. $\Rightarrow \pi_1(X, x_0) \cong \pi_1(Y, \beta(x_0))$.
Theorem 8.3 $\Rightarrow$ the isomorphism classes of the fundamental groups of X and Y do not depend on the choice of basepoints. □

Corollary 8.9. Let X be a contractible space and let $x_0 \in X$. Then

$$\pi_1(X, x_0) = \{1\}.$$

proof. The claim follows immediately from Corollary 8.8 and the fact that the fundamental group of a one-point space is trivial. □

Definition 8.10 A topological space X is called simply connected, if it is path connected and $\pi_1(X, x_0) = \{1\}$ for every $x_0 \in X$.

Notice: Some authors do not require simply connected space to be path connected. They call a space simply connected if every path component of the space is simply connected in our sense.

Corollary 8.11. Assume $\beta : (X, x_0) \to (Y, y_0)$ is (freely) nullhomotopic. Then the induced homomorphism $\beta_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is trivial.

proof. Let $k : X \to Y$, $x \mapsto y_1$, be a constant map. Then

$$k_* : \pi_1(X, x_0) \to \pi_1(Y, y_1),\ [\alpha] \mapsto [k \circ \alpha],$$

is trivial ($k \circ \alpha$ is a constant path at y_1). Assume $\beta : (X, x_0) \to (Y, y_0)$ is nullhomotopic, $\beta \simeq k$. Lemma 8.5 $\Rightarrow$ there is an isomorphism ψ, where

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\beta_*"] \arrow[dr, "k_*"'] & \pi_1(Y, y_0) \arrow[d, "\psi"] \\
& \pi_1(Y, y_1)
\end{tikzcd}$$

Then $\psi \beta_* = k_* \Rightarrow \beta_* = \psi^{-1} k_*$ is trivial.

$\square$

9. The Fundamental Group of a Circle

The exponential map $p: \mathbb{R} \to S^1$ is

$$p(t) = (\underbrace{\cos 2\pi t, \sin 2\pi t)}_{\in \mathbb{R}^2} = \underbrace{e^{i2\pi t}}_{\in \mathbb{C}}$$

It is a continuous surjection and a group homomorphism $(\mathbb{R}, +) \to (S^1, \cdot)$,

$$p(t+s) = e^{i2\pi(t+s)} = e^{i2\pi t} \cdot e^{i2\pi s} = p(t)p(s).$$

$\operatorname{Ker}(p) = p^{-1}(1) = \mathbb{Z}$.

The restriction $p| : (-\frac{1}{2}, \frac{1}{2}) \to S^1 - \{-1\}$ is a homeomorphism. We denote the inverse function of $p|$ by $lg : S^1 - \{-1\} \to (-\frac{1}{2}, \frac{1}{2})$. For example, $p(0) = 1 \Rightarrow lg(1) = 0$.

<u>Definition 9.1.</u> Let X be a topological space and let $f: X \to S^1$ be a function. A continuous function $f': X \to \mathbb{R}$ is called a <u>lift</u> of f, if $p \circ f' = f$.

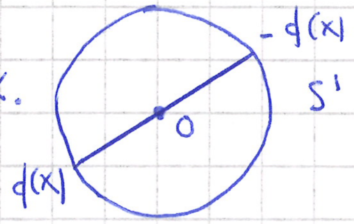
<u>Definition 9.2.</u> Let $A \subset \mathbb{R}^n$ and $a \in A$. We say that A is star-like at a, if the line segment connecting x and a is in A for all $x \in A$.

<u>Proposition 9.3.</u> Let $X \subset \mathbb{R}^n$, $0 \in X$. Assume X is compact and star-like at 0. If $f: X \to S^1$ is a continuous map and $t_0 \in \mathbb{R}$ satisfies $p(t_0) = f(0)$, then f has a unique lift $f': X \to \mathbb{R}$ satisfying $f'(0) = t_0$.

proof:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "f'"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

Since X is compact and $f: X \to S^1$ is continuous, it follows that f is uniformly continuous. Then $\exists\ \delta > 0 : \|x - x'\| < \delta \Rightarrow \|f(x) - f(x')\| < 2$. Hence $f(x') \neq -f(x)$ if $\|x - x'\| < \delta$.



X bounded $\Rightarrow \exists\ n \in \mathbb{N} : \frac{\|x\|}{n} < \delta\ \forall x \in X$.
Then

$$\left\| \frac{(j+1)x}{n} - \frac{jx}{n} \right\| = \frac{\|x\|}{n} < \delta \quad \forall x \in X,\ 0 \leq j < n.$$

X star-like at $0 \Rightarrow \frac{jx}{n}, \frac{(j+1)}{n}x \in X \quad \forall x \in X\ \forall j \in \{0, \ldots, n\}$.

$$\Rightarrow \left\| f\left(\tfrac{j+1}{n}x\right) - f\left(\tfrac{j}{n}x\right) \right\| < 2 \quad \forall x \in X,\ 0 \leq j < n.$$

It follows that the function

$$g_j : X \to S^1 - \{-1\},\ x \mapsto \frac{f\left(\frac{j+1}{n}x\right)}{f\left(\frac{j}{n}x\right)},$$

is well defined and continuous, $0 \leq j < n$.

Write f as a product of functions:

$$f(x) = f(0)\, g_0(x)\, g_1(x) \cdots g_{n-1}(x).$$

Define $f' : X \to \mathbb{R}$ by

$$f'(x) = l_0 + lg(g_0(x)) + \cdots + lg(g_{n-1}(x)).$$

Then ϕ' is continuous. It is a lift of ϕ:

$$p(\phi'(x)) = p[t_0 + lg(g_0(x)) + \dots + lg(g_{n-1}(x))]$$

$$\overset{\text{p homom.}}{=} p(t_0)\, p(lg(g_0(x))) \cdots p(lg(g_{n-1}(x)))$$

$$= \underbrace{p(t_0)}_{\phi(0)}\, g_0(x) \cdots g_{n-1}(x) \quad = \phi(x).$$

Also: $\phi'(0) = t_0 + lg(\underbrace{g_0(0)}_{=1}) + \dots + lg(\underbrace{g_{n-1}(0)}_{=1})$

$$= t_0 + lg(1) + \dots + lg(1)$$

$$= t_0 \,, \quad \text{since } lg(1) = 0.$$

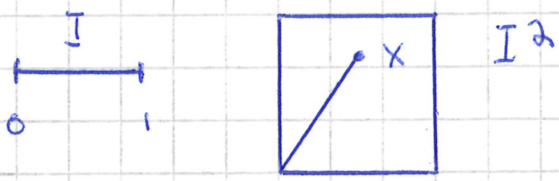
<u>Uniqueness:</u> Assume $\phi', \phi'' : X \to \mathbb{R}$ are lifts of ϕ satisfying $\phi'(0) = \phi''(0) = t_0$. Let $h : X \to \mathbb{R}$, $x \mapsto \phi'(x) - \phi''(x)$. Then

$$p(h(x)) = p(\phi'(x) - \phi''(x)) = \frac{p(\phi'(x))}{p(\phi''(x))} \quad (\text{p homom.})$$

$$= \frac{\phi(x)}{\phi(x)} = 1, \text{ for all } x \in X.$$

Thus $h(X) \subseteq p^{-1}(1) = \mathbb{Z}$. Since X is star-like, it is path connected. $\Rightarrow$ $h(X)$ is a pathconnected subset of $\mathbb{Z}$ $\Rightarrow$ $h(X)$ is a one point set. Now, $h(0) = \phi'(0) - \phi''(0) = t_0 - t_0 = 0$. Thus $h(x) = 0 \;\forall\, x \in X$, i.e., $\phi'(x) = \phi''(x) \;\forall\, x \in X$. $\square$

<u>Notice:</u> I and $I^2 = I \times I$ are star-like at origin.



Corollary 9.4.

1) (uniqueness of path lifting) Every path $\alpha : I \to S^1$ with $\alpha(0) = 1$ has a unique lift $\alpha' : I \to \mathbb{R}$ with $\alpha'(0) = 0$.

2) Let $\alpha, \beta : I \to S^1$ be paths with $\alpha(0) = \beta(0) = 1$. Let $\alpha', \beta' : I \to \mathbb{R}$ be the lifts of α and β, respectively, with $\alpha'(0) = \beta'(0) = 0$. Every homotopy $F : \alpha \simeq \beta$ rel $\dot{I}$ has a unique lift $F' : \alpha' \simeq \beta'$ rel $\dot{I}$. (uniqueness for lifting homotopies).

proof.

1) Since $\alpha(0) = 1 = p(0)$, the claim follows immediately from Proposition 9.3.

2) The homotopy F satisfies $F(0,0) = 1$. Proposition 9.3 $\Rightarrow$ F has a unique lift $F' : I^2 \to \mathbb{R}$ with $F'(0,0) = 0$. The map $s \mapsto F'(s,0)$ is a lift of the map $s \mapsto F(s,0) = \alpha(s)$ and $F'(0,0) = 0$. Uniqueness of the lift of α (part 1) $\Rightarrow$ $F'(s,0) = \alpha'(s)$ $\forall s \in I$.

The map $t \mapsto F'(0,t)$ is a lift of the map $t \mapsto F(0,t) = 1$. Thus it maps I to $p^{-1}(1) = \mathbb{Z}$, which implies that it is a constant map. Then $F'(0,0) = 0 \Rightarrow F'(0,t) = 0$ $\forall t \in I$.

Similarly: $F'(s,1) = \beta'(s)$ $\forall s \in I$ and $F'(1,t)$ is constant.

Thus: $F' : \alpha' \simeq \beta'$ rel $\dot{I}$. $\square$

Definition 9.5. Let $\alpha: I \to S^1$ be a closed path with $\alpha(0) = \alpha(1) = 1$. Let $\alpha': I \to \mathbb{R}$ be the unique lift of α with $\alpha'(0) = 0$. We call $\alpha'(1) \in \mathbb{Z}$ the <u>degree</u> of α and denote it ~~by~~ $\deg \alpha$.

Notice: $p(\alpha'(1)) = \alpha(1) = 1 \Rightarrow \alpha'(1) \in \ker p = \mathbb{Z}$.

If $\alpha \simeq \beta$ rel $\dot{I}$, then $\alpha' \simeq \beta'$ rel $\dot{I}$. Thus

$$\deg \alpha = \alpha'(1) = \beta'(1) = \deg \beta.$$

Thus we obtain a function

$$\deg: \pi_1(S^1, 1) \to \mathbb{Z}, \quad [\alpha] \mapsto \deg \alpha$$

Proposition 9.6. The function

$$\deg: \pi_1(S^1, 1) \to \mathbb{Z}$$

is a group isomorphism.

<u>proof:</u> Three parts: 1) deg is an injection
2) deg is a surjection
3) deg is a group homomorphism.

1) Assume $\deg[\alpha] = \deg[\beta]$. Let α' and β' be the lifts of α and β, respectively. Then $\alpha'(1) = \beta'(1)$ and $\alpha'(0) = \beta'(0) = 0$. Let

$$F: I \times I \to \mathbb{R}, \quad (s,t) \mapsto (1-t)\alpha'(s) + t\beta'(s).$$

Then $F: \alpha' \simeq \beta'$ rel $\dot{I}$. $\Rightarrow pF: \alpha \simeq \beta$ rel $\dot{I}$

$\Rightarrow [\alpha] = [\beta]$

$\therefore$ deg is an injection