Homotopy theory
Exercise 4 (1.10.2015)

1. Let X be a space and $A \subset X$. We say that A is a *deformation retract* of X if there exists a homotopy $H\colon X \times I \to X$ such that

$$H(x,0) = x \quad \text{for every } x \in X,$$

$$H(x,1) \in A \quad \text{for every } x \in X$$

and

$$H(a,1) = a \quad \text{for every } a \in A.$$

We say that A is a *strong deformation retract* of X if there exists a homotopy $H\colon X \times I \to X$ such that

$$H(x,0) = x \quad \text{for every } x \in X,$$

$$H(x,1) \in A \quad \text{for every } x \in X$$

and

$$H(a,t) = a \quad \text{for every } a \in A, t \in I.$$

a) Prove that if A is a deformation retract of X, then A is a retract of X.

b) Prove that if A is a deformation retract of X, then $A \simeq X$.

c) Prove that S^{n-1} is a strong deformation retract of $\mathbb{R}^n \setminus \{0\}$.

d) Give an example where A is a deformation retract of X, but not a strong deformation retract.

e) Give an example where A is a retract of X, but not a deformation retract.

2. Suppose X is a topological space, $x_0 \in X$ and $f, g\colon X \to S^1$ maps for which $f(x_0) = g(x_0)$ and $f \simeq g$. Prove that $f \simeq g$ rel x_0.
[Hint: If $h\colon f \simeq g$, then search for a new homotopy of the form $H_t = u_t \circ h_t$, where u_t is a suitable rotation of the circle.]

3. Prove: If $n \geq 3$, then $\mathbb{R}^n$ and $\mathbb{R}^2$ are not homeomorphic with each other.
[Hint: Remove one point from the sets.]

4. Let $f\colon \bar{B}^2 \to \bar{B}^2$ be a homeomorphism. Prove that $fB^2 = B^2$ and $fS^1 = S^1$. Deduce from this that $\bar{B}^2$ is not a homogeneous space.
[Hint: Remove one point.]
A space X is called *homogeneous*, if given any $a, b \in X$ there exists a homeomorphism $f\colon X \to X$ such that $f(a) = b$.

5. a) Prove that there doesn't exist an embedding $S^1 \to \mathbb{R}$.
b) Prove that there doesn't exist an embedding $S^2 \to \mathbb{R}^2$.

6. Suppose (X, d) is a metric space and $a, b \in X$. Equip the set $P = \Omega(X, a, b)$ with the sup-metric. Prove that two paths $\alpha, \beta \in P$ are path homotopic if and only if they belong to the same path component of P.

Homotopiateoria
Harjoitus 4 (1.10.2015)

1. Olkoon X avaruus ja $A \subset X$. Sanomme, että A on X:n *deformaatioretrakti*, jos on olemassa homotopia $H\colon X \times I \to X$ siten, että

$$H(x,0) = x \;\text{ jokaisella } x \in X,$$
$$H(x,1) \in A \;\text{ jokaisella } x \in X$$

ja

$$H(a,1) = a \;\text{ jokaisella } a \in A.$$

Sanomme, että A on X:n *vahva deformaatioretrakti*, jos on olemassa homotopia $H\colon X \times I \to X$ siten, että

$$H(x,0) = x \;\text{ jokaisella } x \in X,$$
$$H(x,1) \in A \;\text{ jokaisella } x \in X$$

ja

$$H(a,t) = a \;\text{ jokaisella } a \in A, t \in I.$$

a) Osoita: jos A on X:n deformaatioretrakti, niin A on X:n retrakti.

b) Osoita: jos A on X:n deformaatioretrakti, niin $A \simeq X$.

c) Osoita, että S^{n-1} on avaruuden $\mathbb{R}^n \setminus \{0\}$ vahva deformaatioretrakti.

d) Anna esimerkki tilanteesta, jossa A on X:n deformaatioretrakti, mutta ei vahva deformaatioretrakti.

e) Anna esimerkki tilanteesta, jossa A on X:n retrakti, mutta ei deformaatioretrakti.

2. Väisälä, s. 155, tehtävä 21:16

3. Väisälä, s. 175, tehtävä 24:6

4. Väisälä, s. 175, tehtävä 24:7

5. a) Osoita, että ei ole olemassa upotusta $S^1 \to \mathbb{R}$.
b) Osoita, että ei ole olemassa upotusta $S^2 \to \mathbb{R}^2$.

6. Väisälä, s. 159, tehtävä 22:6