Homotopy theory
Exercise 3 (24.9.2015)

1. If σ is a path between points $x_0, x_1 \in X$, then σ induces an isomorphism

$$\sigma_\sharp\colon \pi(X,x_0) \to \pi(X,x_1), \quad \sigma_\sharp(\bar\alpha) = \overline{\sigma^{\leftarrow}}\bar\alpha\bar\sigma.$$

Suppose that $f, g\colon X \to Y$ are homotopic, but not necessarily rel x_0. If we denote $y_0 = f(x_0)$ and $y_1 = g(x_0)$ we have the induced homomorphisms

$$f_*\colon \pi(X,x_0) \to \pi(Y,y_0),\ \ g_*\colon \pi(X,x_0) \to \pi(Y,y_1).$$

Prove the following connection between f_* and g_*: Let $h\colon f \simeq g$. The formula $\sigma(t) = h(x_0,t)$ defines a path in Y from y_0 to y_1. Then

$$g_* = \sigma_\sharp \circ f_*.$$

[Hint: Let $\alpha \in \Omega(X,x_0)$. We should prove that $g_*(\bar\alpha) = \overline{\sigma^{\leftarrow}} f_*(\bar\alpha)\bar\sigma$, that is, $\overline{g\circ\alpha^{\leftarrow}\sigma^{\leftarrow} f\circ\alpha\sigma} = \bar\epsilon$. Let $z_0 = (1,1)$ and $\omega \in \Omega(I^2, z_0)$ the composite of four line paths, which goes around the boundary of I^2 counter-clockwise. Denote $F\colon I^2 \to Y,\ F(s,t) = h(\alpha(s),t)$. Deduce that $\omega \sim \epsilon$ in I^2 which implies that $F \circ \omega \sim \epsilon$ in Y which implies the claim.]

2. With help of the previous exercise, prove the following: If $f\colon X \to Y$ is a homotopy equivalence then $f_*\colon \pi(X,x_0) \to \pi(Y, f(x_0))$ is an isomorphism.

3. Give an example of a covering map $p\colon (X,x_0) \to (Y,y_0)$ and paths $\alpha,\beta \in \Omega(Y,y_0)$ such that $\tilde\alpha(1) = \tilde\beta(1)$, but $\alpha \not\sim \beta$.

4. a) Prove that a covering map is always a surjective open immersion.
[A map $f\colon X \to Y$ is an *embedding*, if the function $f_1\colon X \to fX,\ f_1(x) = f(x)$, is a homeomorphism. A map $f\colon X \to Y$ is an *immersion*, if each point $x \in X$ has a neighbourhood U such that $f|U$ is an embedding.]
b) Prove that a covering map is always an identification map.

5. Suppose (X,x_0) and (Y,y_0) are path connected pointed spaces and $p\colon X \times Y \to X,\ q\colon X \times Y \to Y$ are the projections. Prove that the function

$$(p_*,q_*)\colon \pi(X\times Y,(x_o,y_0)) \to \pi(X,x_0) \times \pi(Y,y_0)$$

is an isomorphism of groups. The group operation in the product of groups is given by $(a, b)(a', b') = (aa', bb')$.
In case the groups are Abelian, the product is usually called the direct sum, and is denoted by $\pi(X, x_0) \oplus \pi(Y, y_0)$.

6. Find examples of spaces $(X, x_0), (Y, y_0)$ and functions $f\colon (X, x_0) \to (Y, y_0)$ such that
1) f is injective, f_* is not injective
2) f is not injective, f_* is injective
3) f is surjective, f_* is not surjective
4) f is not surjective, f_* is surjective.

Homotopiateoria
Harjoitus 3 (24.9.2015)

1. Väisälä, s. 166, tehtävä 23:2 (Lue ensin kohta 23.17, s. 165)

2. Väisälä, s. 166, tehtävä 23:3

3. Anna esimerkki peitekuvauksesta $p\colon (X, x_0) \to (Y, y_0)$ ja poluista $\alpha, \beta \in \Omega(Y, y_0)$ siten, että $\tilde{\alpha}(1) = \tilde{\beta}(1)$, mutta $\alpha \not\simeq \beta$.

4. a) Osoita, että peitekuvaus on aina surjektiivinen avoin immersio (V:24.2.1, s. 168). Erityisesti siis peitekuvaus on aina jatkuva.

b) Osoita, että peitekuvaus on aina samastuskuvaus.

5. Väisälä, s. 166, tehtävä 23:4

6. Keksi esimerkkejä avaruuksista $(X, x_0), (Y, y_0)$ ja funktioista $f\colon (X, x_0) \to (Y, y_0)$ siten, että
1) f on injektio, f_* ei ole injektio
2) f ei ole injektio, f_* on injektio
3) f on surjektio, f_* ei ole surjektio
4) f ei ole surjektio, f_* on surjektio.