HARMONIC ANALYSIS AND SQUARE FUNCTIONS: EXERCISE SET 2

Return your written solutions preferably directly to Emil Vuorinen (office C435, emil.vuorinen@helsinki.fi). You can also return them during the lectures. **Deadline for the second set is Friday, November 20.** There is a separate file with some hints provided. If you need additional guidance you can email Emil to schedule a meeting, or ask during the lectures.

(1) Let μ be a Radon measure in $\mathbb{R}^n$. Show that for every cube Q there holds that

$$\int_Q |f| \, d\mu \le \frac{s}{s-1} \mu(Q)^{1-1/s} \|f\|_{L^{s,\infty}(\mu)}, \qquad s > 1,$$

where, as usual, we set

$$\begin{aligned} \|f\|_{L^{s,\infty}(\mu)} &:= \inf\{C > 0 \colon \ \mu(\{|f| > \lambda\}) \le \frac{C^s}{\lambda^s} \text{ for all } \lambda > 0\} \\ &= \sup_{\lambda > 0} \lambda \mu(\{|f| > \lambda\})^{1/s}. \end{aligned}$$

(2) Let μ be a Radon measure in $\mathbb{R}^n$. Define the following variant of the maximal function

$$M_{\mu,p} f := [M_\mu(|f|^p)]^{1/p}, \qquad p > 1.$$

Show that

$$\frac{1}{\mu(Q)} \int_Q M_{\mu,p}(f 1_Q) \, d\mu \le C \Big(\frac{1}{\mu(Q)} \int_Q |f|^p \, d\mu \Big)^{1/p}.$$

(3) Let μ be a measure of order m in $\mathbb{R}^n$, $Q \subset \mathbb{R}^n$ be a cube and R be the *smallest* $(6, 6^{m+1})$-μ-doubling cube of the form $6^k Q$, $k \ge 0$. Show that

$$\int_{R \setminus Q} \frac{d\mu(x)}{|x - c_Q|^m} \le C,$$

where $C < \infty$ is independent of Q and R.

(4) Show that the following claim made during the proof of Theorem 3.13 holds: for every $x \in \mathbb{R}^n \setminus 2Q_i$ we have

$$V_\nu w_i(x) \le C \frac{|\nu|(Q_i)}{|x - c_{Q_i}|^m}.$$

The following exercises deal with dyadic analysis. Here $\mathcal{D}_0$ is the collection of standard dyadic cubes in $\mathbb{R}^n$ i.e.

$$\mathcal{D}_0 = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_0^k, \qquad \mathcal{D}_0^k = \{2^{-k}([0,1)^n + m) \colon\ m \in \mathbb{Z}^n\}.$$

The main properties of dyadic cubes are that each $\mathcal{D}_0^k$ is a partition (a pairwise disjoint cover) of $\mathbb{R}^n$ and each $\mathcal{D}_0^{k+1}$ is a refinement of the previous $\mathcal{D}_0^k$ (cubes from $\mathcal{D}_0^k$ can be written as the disjoint union of their 2^n dyadic children from $\mathcal{D}_0^{k+1}$). Moreover, maximal dyadic cubes with respect to some property are always disjoint.

(5) Let σ and μ be Radon measures in $\mathbb{R}^n$. Define

$$\underline{D}_{\mathcal{D}_0}(\sigma, \mu, x) = \liminf_{k\to\infty} \frac{\sigma(R_k(x))}{\mu(R_k(x))},$$

where $R_k(x)$ denotes the unique dyadic cube $R \in \mathcal{D}_0^k$ for which $x \in R$. Suppose $\lambda > 0$ and that $A \subset \mathbb{R}^n$ is a Borel set so that

$$\underline{D}_{\mathcal{D}_0}(\sigma, \mu, x) \le \lambda$$

for all $x \in A$. Show that

$$\sigma(A) \le \lambda\mu(A).$$

(6) Let μ be a Radon measure in $\mathbb{R}^n$. Suppose also that for each $Q \in \mathcal{D}_0$ we are given a function A_Q satisfying that $\operatorname{spt} A_Q \subset Q$. Define the square function

$$Af(x) := \Big[\sum_{Q\in\mathcal{D}_0} |\langle f\rangle_Q^\mu|^2 |A_Q(x)|^2 \Big]^{1/2},$$

where

$$\langle f\rangle_Q^\mu := \frac{1}{\mu(Q)} \int_Q f \, d\mu.$$

Let $p \in (1, 2]$. Show that

$$\|Af\|_{L^p(\mu)} \le C \cdot \mathrm{Car}_p((A_Q)_{Q\in\mathcal{D}_0}) \|f\|_{L^p(\mu)},$$

where

$$\mathrm{Car}_p((A_Q)_{Q\in\mathcal{D}_0}) := \Big(\sup_{R\in\mathcal{D}_0} \frac{1}{\mu(R)} \int_R \Big[\sum_{\substack{Q\in\mathcal{D}_0 \\ Q\subset R}} |A_Q(x)|^2 \Big]^{p/2} d\mu(x) \Big)^{1/p}.$$

(7) Let μ be a Radon measure in $\mathbb{R}^n$. Suppose that for every $Q \in \mathcal{D}_0$ we are given a function φ_Q so that $\operatorname{spt} \varphi_Q \subset Q$ and φ_Q is constant on the dyadic children of Q i.e.

$$\varphi_Q = \sum_{Q'\in \mathrm{ch}(Q')} c_{Q'} 1_{Q'}$$

for some constants $c_{Q'}$. Assume also that $\|\varphi_Q\|_{L^\infty(\mu)} \leq 1$. For every $P \in \mathcal{D}_0$ define

$$\Phi_P := \sup_{\epsilon>0} \Big| \sum_{\substack{Q\in\mathcal{D}_0 \colon Q\subset P \\ \ell(Q)>\epsilon}} \varphi_Q \Big|.$$

Suppose that for every $P \in \mathcal{D}_0$ we have

$$\mu(\{x \in P \colon \Phi_P(x) > 1\}) \leq \mu(P)/2. \tag{0.1}$$

Show that for every $P \in \mathcal{D}_0$ and $t > 1$ we have

$$\mu(\{x \in P \colon \Phi_P(x) > t\}) \leq 2^{-(t-1)/2}\mu(P).$$

(8) Show that in the situation of the previous exercise we have for every $P \in \mathcal{D}_0$ that

$$\int_P [\Phi_P(x)]^p \, d\mu(x) \leq C\mu(P), \qquad 0 < p < \infty.$$

That is, the uniform L^0-condition (0.1) implies integrability for all $p > 0$!