EXERCISE SET 3, SUGGESTIONS FOR SOLUTIONS

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Exercise 1. Here we check that for any $f \in L^1_{\text{loc}}(\sigma)$ the martingale differences $\Delta_P f, P \in \mathcal{D}^{tr}$, defined in the lecture notes, satisfy

$$\int \Delta_P f d\sigma = 0.$$

Indeed,

$$\begin{aligned}
\int \Delta_P f d\sigma &= \sum_{P' \in \text{ch}(P) \cap \mathcal{D}^{tr}} \Big(\frac{\langle f \rangle_{P'}}{\langle b \rangle_{P'}} - \frac{\langle f \rangle_P}{\langle b \rangle_P} \Big) \int_{P'} b d\sigma \\
&+ \sum_{P' \in \text{ch}(P) \cap \mathcal{D}^{term}} \Big(\int_{P'} f d\sigma - \frac{\langle f \rangle_P}{\langle b \rangle_P} \int_{P'} b d\sigma \Big) \\
&= \sum_{P' \in \text{ch}(P) \cap \mathcal{D}^{tr}} \Big(\int_{P'} f d\sigma - \frac{\langle f \rangle_P}{\langle b \rangle_P} \int_{P'} b d\sigma \Big) \\
&+ \sum_{P' \in \text{ch}(P) \cap \mathcal{D}^{term}} \Big(\int_{P'} f d\sigma - \frac{\langle f \rangle_P}{\langle b \rangle_P} \int_{P'} b d\sigma \Big) \\
&= \int_P f d\sigma - \frac{\langle f \rangle_P}{\langle b \rangle_P} \int_P b d\sigma = 0.
\end{aligned}$$

Exercise 2. Let the assumptions be as in the statement of the exercise. To find the set $G \subset E$ with positive measure where the square function is bounded, we want to use the big pieces global Tb with the test function 1_Q. If one defines the collections T_ω with the function 1_Q and the constant $c_{acc} = \frac{1}{2}$ as in the lecture notes, one gets $T_\omega = \emptyset$. Indeed, for every $R \in \mathcal{D}(\omega)$ it holds that

$$\frac{\int_R 1_Q d\sigma}{\sigma(R)} = \frac{\sigma(R)}{\sigma(R)} = 1,$$

where we used the fact that σ is supported on Q.

Hence, to apply the theorem, we want to find a set $H \supset Q \setminus E$ and constants $s, C_0, C_1 > 0$ so that the following properties hold:

- $\sigma(H) \leq \delta_0 \sigma(Q)$ for some $\delta \in (0, 1)$.
- If $B_r \subset \mathbb{R}^n$ is a ball of radius r and $\sigma(B_r) > C_0 r^m$, then $B_r \subset H$.
- $\sup_{\lambda > 0} \lambda^s \sigma(\{x \in Q \setminus H : V_{\sigma,Q} 1_Q(x) > \lambda\}) \leq C_1 \sigma(Q)$.

If such a set H is found, then the big pieces global Tb gives us a set $G \subset Q \setminus H \subset E$ with positive measure where the L^2-bound for the square function holds.

Let us now construct the set H. Write $\varphi(x) := \sup_{r>0} \frac{\sigma(B(x,r))}{r^m}$. Set first

$$H_0 := \{x \in \mathbb{R}^n : \varphi(x) > \lambda_0\}$$

for some λ_0 to be specified later. For every $x \in H_0$ define

$$R(x) := \sup\{r > 0 : \frac{\sigma(B(x,r))}{r^m} > \lambda_0\},$$

and then write

$$H_1 := \bigcup_{x \in H_0} B(x, R(x)).$$

If $y \in B(x, R(x))$ for some $x \in H_0$, then

$$\frac{\sigma(B(y, 2R(x))}{(2R(x))^m} \geq 2^{-m} \frac{\sigma(B(x, R(x))}{R(x)^m} \geq 2^{-m}\lambda_0.$$

Hence $H_1 \subset \{x \in \mathbb{R}^n : \varphi(x) \geq 2^{-m}\lambda_0\}$. Because $\varphi(x) < \infty$ for all $x \in E$, we see that $\sigma(E \cap H_1) \leq \frac{1}{3}\sigma(E)$ if λ_0 is large enough. Fix now one such λ_0.

Next, consider the set $S_0 := \{x \in E : V_{\sigma,Q}1_Q(x) > t_0\}$ for some $t_0 > 0$. Since $V_{\sigma,Q}1_Q(x) < \infty$ for all $x \in E$, we get $\sigma(S_0) \leq \frac{\sigma(E)}{3}$ for some big enough t_0. Fix one such t_0.

Now, finally set $H := H_1 \cup S_0 \cup (Q \setminus E)$. If $\frac{\sigma(B(x,r))}{r^m} > \lambda_0$, then $B(x,r) \subset B(x, R(x)) \subset H_1 \subset H$. Also, we have

$$\sup_{\lambda>0} \lambda\sigma(\{x \in Q \setminus H : V_{\sigma,Q}1(x) > \lambda\}) \leq t_0\sigma(Q).$$

Thus, we have verified the required properties for the set H with $\delta_0 = \frac{\sigma(Q)-\frac{1}{3}\sigma(E)}{\sigma(Q)} < 1, C_0 = \lambda_0, C_1 = t_0$ and $s = 1$. This finishes the proof.

Exercise 3. Let $(s_t)_{t>0}$ be an m-LP family and $\theta_t^\mu, t > 0$, the corresponding integral operators. Suppose μ is a Radon measure on $\mathbb{R}^n$ and $Q \subset \mathbb{R}^n$ is a cube. The claim is that

$$\Big\| x \mapsto 1_Q(x)\Big(\int_{l(Q)}^\infty |\theta_t^\mu f(x)|^2 \frac{dt}{t}\Big)^{\frac{1}{2}}\Big\|_{L^2(\mu)} \lesssim \frac{\mu(Q)}{l(Q)^m}\|f\|_{L^2(\mu)}$$

for every f with $\text{spt} f \subset Q$.

To prove this, let $f \in L^2(\mu)$ with support in Q. Then, for every $x \in \mathbb{R}^n$, we have

$$\int_{l(Q)}^\infty |\theta_t^\mu f(x)|^2 \frac{dt}{t} \lesssim \int_{l(Q)}^\infty \frac{\|f\|_{L^1(\mu)}^2}{t^{2m}} \frac{dt}{t} \lesssim \frac{\mu(Q)\|f\|_{L^2(\mu)}^2}{l(Q)^{2m}}.$$

Integrating this over $x \in Q$ proves the claim.

Exercise 4. Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are two dyadic lattices in $\mathbb{R}^n$. Let $\gamma \in (0,1)$ and $r = 1, 2, \ldots$ We say that a cube $R \in \mathcal{D}_1$ is $(\gamma, r) - \mathcal{D}_2$-good if $d(R, \partial Q) > l(R)^\gamma l(Q)^{(1-\gamma)}$ for all cubes $Q \in \mathcal{D}_2$ with $l(Q) \geq 2^r l(R)$. Let $\mathcal{D}_{1,\text{good}}$ be the collection of these good cubes.

Also, suppose that μ is a Radon measure in $\mathbb{R}^n$, and let $M > 1$ be fixed. For any function $a \in \text{BMO}_M^2(\mu)$ we define the operator Π_a by

$$\Pi_a f := \sum_{R \in \mathcal{D}_2} \langle f \rangle_R \sum_{\substack{Q \in \mathcal{D}_{1,\text{good}} \\ Q \subset R \\ l(Q) = 2^{-r} l(R)}} D_Q a, \qquad f \in L^2(\mu). \tag{0.1}$$

The claim is that, for a big enough r, there exists an absolute constant C such that

$$\|\Pi_a f\|_{L^2(\mu)} \le C\|a\|_{\mathrm{BMO}^2_M(\mu)}\|f\|_{L^2(\mu)} \tag{0.2}$$

holds for all $f \in L^2(\mu)$ and $a \in \mathrm{BMO}^2_M(\mu)$.

Proof. Let $\tilde{\mathcal{D}}_2 \subset \mathcal{D}_2$ be an arbitrary finite subcollection. Then

$$\begin{aligned}\Big\| \sum_{R\in\tilde{\mathcal{D}}_2} \langle f\rangle_R \sum_{\substack{Q\in\mathcal{D}_{1,\mathrm{good}}\\ Q\subset R\\ l(Q)=2^{-r}l(R)}} D_Q a\Big\|^2_{L^2(\mu)} &= \sum_{R\in\tilde{\mathcal{D}}_2} |\langle f\rangle_R|^2 \sum_{\substack{Q\in\mathcal{D}_{1,\mathrm{good}}\\ Q\subset R\\ l(Q)=2^{-r}l(R)}} \|D_Q a\|^2_{L^2(\mu)}\\ &\le \sum_{R\in\mathcal{D}_2} |\langle f\rangle_R|^2 \sum_{\substack{Q\in\mathcal{D}_{1,\mathrm{good}}\\ Q\subset R\\ l(Q)=2^{-r}l(R)}} \|D_Q a\|^2_{L^2(\mu)}.\end{aligned}$$

We will show that there exists a constant C such that

$$\sum_{R\in\mathcal{D}_2} |\langle f\rangle_R|^2 \sum_{\substack{Q\in\mathcal{D}_{1,\mathrm{good}}\\ Q\subset R\\ l(Q)=2^{-r}l(R)}} \|D_Q a\|^2_{L^2(\mu)} \le C\|a\|^2_{\mathrm{BMO}^2_M(\mu)}\|f\|^2_{L^2(\mu)}. \tag{0.3}$$

From this it follows that Π_a is well defined, that is, the series in (0.1) converges in $L^2(\mu)$, and that the L^2-bound (0.2) holds.

From (0.3) we see that it is enough to verify the Carleson property for the numbers

$$a_R := \sum_{\substack{Q\in\mathcal{D}_{1,\mathrm{good}}\\ Q\subset R\\ l(Q)=2^{-r}l(R)}} \|D_Q a\|^2_{L^2(\mu)}, \quad R\in\mathcal{D}_2. \tag{0.4}$$

In other words, we want to have a constant C such that for all $R_0 \in \mathcal{D}_2$ it holds that

$$\sum_{\substack{R\in\mathcal{D}_2\\ R\subset R_0}} a_R \le C\mu(R_0).$$

To this end, fix some cube $R_0 \in \mathcal{D}_2$. For any $Q \in \mathcal{D}_2$ denote by $\mathcal{W}(Q)$ the collection of maximal cubes $Q' \in \mathcal{D}_1$ such that $Q' \subset Q$, $l(Q') \le 2^{-r}l(Q)$ and $d(Q',\partial Q) > l(Q')^\gamma l(Q)^{1-\gamma}$. Using this we get

$$\begin{aligned}\sum_{\substack{R\in\mathcal{D}_2\\ R\subset R_0}} a_R = \sum_{\substack{Q\in\mathcal{D}_{1,\mathrm{good}}\\ Q\subset R_0\\ l(Q)\le 2^{-r}l(R)}} \|D_Q a\|^2_{L^2(\mu)} &\le \sum_{R\in\mathcal{W}(R_0)} \sum_{\substack{Q\in\mathcal{D}_1\\ Q\subset R}} \|D_Q a\|^2_{L^2(\mu)}\\ &= \sum_{R\in\mathcal{W}(R_0)} \int_R |a-\langle a\rangle_R|^2 d\mu \le \sum_{R\in\mathcal{W}(R_0)} \|a\|^2_{\mathrm{BMO}^2_M(\mu)}\mu(MR).\end{aligned} \tag{0.5}$$

To conclude the estimate, we would like to have

$$\sum_{R\in\mathcal{W}(R_0)} \mu(MR) \lesssim \mu(R_0).$$

This will follow from the facts that the cubes $MR, R\in\mathcal{W}(R_0)$, have bounded overlap and satisfy $MR \subset U$. We show this next.

Let $R \in \mathcal{W}(R_0)$. Then, by definition, we have $d(R, \partial R_0) > l(R)^\gamma l(R_0)^{1-\gamma} \geq 2^{r(1-\gamma)} l(R)$. This shows that $MR \subset R_0$ if r is big enough. Also, if $l(R) < 2^{-r} l(R_0)$, then by the maximality in the definition of $\mathcal{W}(R_0)$ we have

$$\begin{aligned} d(R, \partial R_0) &\leq \text{diam}(R) + d(R^{(1)}, \partial R_0) \leq \text{diam}(R) + l(R^{(1)})^\gamma l(R_0)^{1-\gamma} \\ &= \text{diam}(R) + 2^\gamma l(R)^\gamma l(R_0)^{1-\gamma} \sim l(R)^\gamma l(R_0)^{1-\gamma}, \end{aligned}$$

since $l(R)^\gamma l(R_0)^{1-\gamma} \geq l(R) \sim \text{diam}(R)$. Hence, we have shown that if $l(R) < 2^{-r} l(R_0)$, then

$$l(R)^\gamma l(R_0)^{1-\gamma} < d(R, \partial R_0) \leq C_1 l(R)^\gamma l(R_0)^{1-\gamma} \tag{0.6}$$

for some constant $C_1 = C_1(\gamma, n)$.

Suppose now $R, R' \in \mathcal{W}(R_0)$ and $MR \cap MR' \neq \emptyset$. The idea is that these cubes must have comparable (depending on M and γ) sidelengths, or otherwise the bigger cube is too close to the boundary ∂R_0. Assume that $l(R) = 2^{-k} l(R')$ for some $k = 1, 2, \ldots$ Then

$$\begin{aligned} d(R', \partial R_0) &\leq M\text{diam}(R') + M\text{diam}(R) + d(R, \partial R_0) \\ &\leq M\text{diam}(R') + M\text{diam}(R) + C_1 l(R)^\gamma l(R_0)^{1-\gamma} \\ &= M\text{diam}(R') + M 2^{-k}\text{diam}(R') + C_1 2^{-k\gamma} l(R')^\gamma l(R_0)^{1-\gamma}. \end{aligned}$$

If $k = k(M, \gamma, n)$ is big enough, then

$$M 2^{-k}\text{diam}(R') + C_1 2^{-k\gamma} l(R')^\gamma l(R_0)^{1-\gamma} \leq \frac{1}{4} l(R')^\gamma l(R_0)^{1-\gamma}.$$

Also, we have

$$\begin{aligned} M\text{diam}(R') &= \frac{n^{\frac{1}{2}} M l(R')}{l(R')^\gamma l(R_0)^{1-\gamma}} l(R')^\gamma l(R_0)^{1-\gamma} \\ &= n^{\frac{1}{2}} M \frac{l(R')^{1-\gamma}}{l(R_0)^{1-\gamma}} l(R')^\gamma l(R_0)^{1-\gamma} \\ &\leq n^{\frac{1}{2}} M 2^{-r(1-\gamma)} l(R')^\gamma l(R_0)^{1-\gamma} \leq \frac{1}{4} l(R')^\gamma l(R_0)^{1-\gamma} \end{aligned}$$

if $r = r(M, \gamma, n)$ is big enough.

Combining the above estimates, we see that for a fixed big enough r, there exists a $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, then

$$d(R', \partial R_0) \leq \frac{3}{4} l(R')^\gamma l(R_0)^{1-\gamma},$$

which is a contradiction because $R' \in \mathcal{W}(R_0)$.

Let $R \in \mathcal{W}(R_0)$. We have shown that if $R' \in \mathcal{W}(R_0)$ and $MR \cap MR' \neq \emptyset$, then $2^{-k_0} l(R') \leq l(R) \leq 2^{k_0} l(R')$. Hence also $R' \subset CR$ for some constant $C = C(M, k_0)$. Since the number of this kind of intervals R' is bounded by some constant depending on the dimension n and $C(M, k_0)$, we finally see that the number of intervals $R' \in \mathcal{W}(R_0)$ such that $MR \cap MR' \neq \emptyset$ is uniformly bounded. $\square$

Exercise 5. Let μ be a Radon measure of order m in $\mathbb{R}^n$ and $p \in [1, \infty)$. Suppose $Q \subset \mathbb{R}^n$ is a cube with t-small boundary. The claim is that

$$\int_Q \Big(\int_{2Q \setminus Q} \frac{d\mu(y)}{|x-y|^m} \Big)^p d\mu(x) \lesssim t\mu(2Q).$$

Proof. For any $x \in \mathbb{R}^n$ and any non negative $k \in \mathbb{Z}$ we have

$$\int_{\bar{B}(x,2l(Q))\setminus B(x,2^{-k}l(Q))} \frac{d\mu(y)}{|x-y|^m} = \sum_{l=0}^{k} \int_{\bar{B}(x,2^{-l+1}l(Q))\setminus B(x,2^{-l}l(Q))} \frac{d\mu(y)}{|x-y|^m}$$
$$\leq \sum_{l=0}^{k} \frac{\mu(\bar{B}(x,2^{-l+1}l(Q)))}{(2^{-l}l(Q))^m} \lesssim k+1.$$

Divide the interior int$Q$ of the cube into subsets $B_k := \{x \in Q : 2^{-k-1}l(Q) \leq d(x,\partial Q) \leq 2^{-k}l(Q)\}, k = 1,2,\dots$ The small boundary property of Q implies that $\mu(\partial Q) = 0$ and $\mu(B_k) \leq t2^{-k}\mu(2Q)$. Hence

$$\int_Q \Big(\int_{2Q\setminus Q} \frac{d\mu(y)}{|x-y|^m} \Big)^p d\mu(x) \leq \sum_{k=1}^{\infty} \int_{B_k} \Big(\int_{\bar{B}(x,2l(Q))\setminus B(x,2^{-k-1}l(Q))} \frac{d\mu(y)}{|x-y|^m} \Big)^p d\mu(x)$$
$$\lesssim \sum_{k=1}^{\infty} (k+2)^p \mu(B_k) \leq \sum_{k=1}^{\infty} (k+2)^p t 2^{-k} \mu(2Q)$$
$$\lesssim t\mu(2Q).$$

□