EXERCISE SET 1, SUGGESTIONS FOR SOLUTIONS

EMIL VUORINEN

Any comments about the exercises or solutions are warmly welcomed!

About notation. If $A, B > 0$ are two numbers we write $A \lesssim B$ to mean that there exists some absolute constant $C > 0$ such that $A \leq CB$, and the constant C does not depend on any relevant information in the situation. For example, knowing that the centred Hardy-Littlewood maximal function M_μ is bounded in $L^p(\mu)$, $1 < p \leq \infty$, we might when we are not interested in precise constants write that $\|M_\mu f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}$, where the implicit constant is the norm of the maximal operator. No confusion should arise about this notation. Also $A \lesssim B \lesssim A$ is abbreviated as $A \sim B$.

Exercise 1. The claim is that if μ is finite or of order m, then for any $f \in \bigcup_{p \in [1,\infty]} L^p(\mu)$ we have

$$\int_{\mathbb{R}^n} |s_t(x,y)f(y)| d\mu(y) < \infty$$

for all $x \in \mathbb{R}^n, t > 0$. So consider some $p \in [1, \infty]$ and $f \in L^p(\mu)$, and suppose first that μ is finite. Then the size estimate for the kernel gives

$$\int_{\mathbb{R}^n} |s_t(x,y)f(y)| d\mu(y) \lesssim \frac{\|f\|_{L^1(\mu)}}{t^m}.$$

Assume then that μ is of order m. Hölder's inequality leads to

$$\int_{\mathbb{R}^n} |s_t(x,y)f(y)| d\mu(y) \lesssim \Big(\int_{\mathbb{R}^n} \frac{t^{\alpha p'}}{(t + |x-y|)^{(m+\alpha)p'}} d\mu(y) \Big)^{\frac{1}{p'}} \|f\|_{L^p(\mu)}$$

with the usual interpretation if $p' = \infty$, in which case we are clearly already done.

To estimate the integral when $p' \neq \infty$, write $(m+\alpha)p' = m + (m+\alpha)p' - m =: m + \beta$, where $\beta > 0$. Next we are going to divide the integration area into certain annuli centred at x. This is an important idea and variations of this will reappear also later in the exercises and during the course.

We have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \frac{1}{(t+|x-y|)^{m+\beta}} d\mu(y) &= \int_{B(x,t)} \frac{1}{(t+|x-y|)^{m+\beta}} d\mu(y) \\
&+ \sum_{k=1}^{\infty} \int_{B(x,2^k t)\setminus B(x,2^{k-1}t)} \frac{1}{(t+|x-y|)^{m+\beta}} d\mu(y) \\
&\le \frac{\mu(B(x,t))}{t^{m+\beta}} + \sum_{k=1}^{\infty} \frac{\mu(B(x,2^k t))}{(2^{k-1}t)^{m+\beta}} \qquad (0.1)\\
&\lesssim \frac{t^m}{t^{m+\beta}} + \sum_{k=1}^{\infty} \frac{(2^k t)^m}{(2^k t)^{m+\beta}} \\
&= \frac{1}{t^\beta} + \frac{1}{t^\beta} \sum_{k=1}^{\infty} \frac{1}{2^{k\beta}} \lesssim \frac{1}{t^\beta}.
\end{aligned}
$$

Note how we used the size of the parameter t in the nominator to set the sizes of the annuli.

Exercise 2. We assume that μ is of order m and show that $|\theta_t^\mu f(x)| \le C M_\mu f(x)$ for all $f \in L^1_{loc}(\mu)$. Use of the size estimate of the kernel and a division into annuli as in Exercise 1 give

$$
\begin{aligned}
|\theta_t^\mu f(x)| &\lesssim \int_{\mathbb{R}^n} \frac{t^\alpha |f(y)|}{(t+|x-y|)^{m+\alpha}} d\mu(y) \\
&\sim t^\alpha \int_{B(x,t)} \frac{|f(y)|}{t^{m+\alpha}} d\mu(y) + \sum_{k=1}^{\infty} \int_{B(x,2^k t)\setminus B(x,2^{(k-1)}t)} \frac{t^\alpha |f(y)|}{(2^k t)^{m+\alpha}} d\mu(y) \\
&\lesssim \frac{t^\alpha}{t^\alpha} \frac{1}{\mu(B(x,t))} \int_{B(x,t)} |f(y)| d\mu(y) \\
&+ \sum_{k=1}^{\infty} \frac{t^\alpha}{2^{k\alpha} t^\alpha} \frac{1}{\mu(B(x,2^k t))} \int_{B(x,2^k t)} |f(y)| d\mu(y) \\
&\le M_\mu f(x) \sum_{k=0}^{\infty} \frac{1}{2^{k\alpha}} \sim M_\mu f(x).
\end{aligned}
$$

Since the *centred* Hardy-Littlewood maximal function is bounded in $L^2(\mu)$ (and also for every $p \in (1,\infty]$), we see that the family $\theta_t, t > 0$ is uniformly bounded in $L^2(\mu)$ (and in $L^p(\mu)$ for every $p \in (1,\infty]$).

To get a family $\{V_{i,\mu}\}_{i\in\mathbb{N}}$ of square functions such that $\|V_{i,\mu}\|_{L^2(\mu)\to L^2(\mu)} \le C(i), V_{i,\mu} f \le V_{i+1,\mu} f$ and $V_{i,\mu} f(x) \to V_\mu f(x)$ as $i \to \infty$ for all $x \in \mathbb{R}^n$, define the kernels $s_{i,t}(x,y) := s_t(x,y) 1_{\frac{1}{i} \le t \le i}$. For all $i \in \mathbb{N}$ the collection $(s_{i,t})_{t>0}$ is clearly an m-LP-family of kernels and with it we can define the corresponding square function $V_{i,\mu}$.

Now directly from the definition we see that, if $f \in \bigcup_{p\in[1,\infty]} L^p(\mu)$, then

$$
V_{i,\mu} f(x) = \Big(\int_{\frac{1}{i}}^{i} |\theta_t^\mu f(x)|^2 \frac{dt}{t} \Big)^{\frac{1}{2}}.
$$

From here we deduce that $V_{i,\mu} f(x) \le V_{i+1,\mu} f(x)$ and $V_{i,\mu} f(x) \to V_\mu f(x)$ as $i \to \infty$. To have the $L^2(\mu)$ boundedness, we use the above formula and the beginning of this exercise to get

$$V_{i,\mu}f(x) \lesssim \Big(\int_{\frac{1}{i}}^{i} \frac{dt}{t} \Big)^{\frac{1}{2}} M_\mu f(x).$$

When dealing with square functions, for example when we want to show that they are bounded with some quantitative constant as later in this course, it can sometimes be nice to know that the square function in question is *a priori* bounded, that is, there exists *some* constant C such that $\|V_\mu f\|_{L^2(\mu)} \leq C\|f\|_{L^2(\mu)}$ for all $f \in L^2(\mu)$. This in particular implies that $\|V_\mu f\|_{L^2(\mu)}$ is a finite number for all $L^2(\mu)$-functions. To achieve this one can in some cases use the above *truncated* square functions $V_{i,\mu}$, show that they satisfy the required L^2 boundedness property and then take the limit $i \to \infty$ to get the conclusion for original operator V_μ.

Exercise 3. In this exercise the task is to show that the vector space $M(\mathbb{R}^n)$ of all complex Borel measures on $\mathbb{R}^n$ is a Banach space when equipped with the total variation norm (one should check that this is first of all a vector space and then that the total variation defines a norm). So let $(\nu_i)_{i=1}^\infty \subset M(\mathbb{R}^n)$ be a Cauchy sequence. Supposing there exists some limit measure ν, then for any Borel set A it holds that

$$|\nu(A) - \nu_i(A)| \leq |\nu - \nu_i|(A) \leq \|\nu - \nu_i\| \to 0,$$

as $i \to \infty$. Thus, if the limit measure exists, then $\lim_{i\to\infty} \nu_i(A) = \nu(A)$. So we try to define the measure ν like this.

Let $\varepsilon > 0$. By the definition of a Cauchy sequence there exists $N_\varepsilon \in \mathbb{N}$ such that $\|\nu_i - \nu_j\| \leq \varepsilon$ for all $i, j \geq N_\varepsilon$. Now for any $i, j \geq N_\varepsilon$ we have

$$|\nu_i(A) - \nu_j(A)| \leq |\nu_i - \nu_j|(A) \leq \|\nu_i - \nu_j\| \leq \varepsilon.$$

Hence for any Borel set A the sequence $(\nu_i(A))_{i=1}^\infty$ is a Cauchy sequence in $\mathbb{C}$, and by completeness of the complex numbers it has a limit as $i \to \infty$.

Now the only way to define the possible limit measure is

$$\nu(A) := \lim_{i\to\infty} \nu_i(A), \quad A \in \text{Bor}(R^n),$$

and this is what we do now. Then the thing to do is to show that ν is a complex Borel measure and that $\nu_i \to \nu$ in the variation norm, as $i \to \infty$.

To this end take arbitrary Borel sets $A, A_k, k = 1, 2, 3, \ldots$, such that $A_k \cap A_l = \emptyset$ if $k \neq l$ and $A = \bigcup_{k=1}^\infty A_k$. Let also $\varepsilon > 0$ be arbitrary. By the definition of a Cauchy sequence choose some $N_\varepsilon \in \mathbb{N}$ such that $\|\nu_i - \nu_j\| \leq \varepsilon$ for all $i, j \geq N_\varepsilon$. Fix any $j_0 \geq N_\varepsilon$. If $l = 1, 2, \ldots$ is any number, then

$$\sum_{k=1}^{l} |\nu(A_k) - \nu_{j_0}(A_k)| = \lim_{i\to\infty} \sum_{k=1}^{l} |\nu_i(A_k) - \nu_{j_0}(A_k)| \leq \limsup_{i\to\infty} \|\nu_i - \nu_{j_0}\| \leq \varepsilon.$$

Since this holds for all l we get

$$\sum_{k=1}^{\infty} |\nu(A_k) - \nu_{j_0}(A_k)| \leq \varepsilon, \tag{0.2}$$

and thus also

$$\sum_{k=1}^{\infty} |\nu(A_k)| \le \varepsilon + \|\nu_{j_0}\|. \tag{0.3}$$

By (0.3) it makes sense [1] to talk about $\sum_k \nu(A_k)$. By (0.2) and the fact that $\nu_i(A) = \sum_k \nu_i(A_k)$ for all i we have

$$\begin{aligned} |\nu(A) - \sum_k \nu(A_k)| &\le |\nu(A) - \nu_{j_0}(A)| + |\sum_k \nu_{j_0}(A_k) - \sum_k \nu(A_k)| \\ &\le |\nu(A) - \nu_{j_0}(A)| + \sum_k |\nu_{j_0}(A_k) - \nu(A_k)| \\ &\le 2\varepsilon, \end{aligned}$$

where we clearly had $|\nu(A) - \nu_{j_0}(A)| = \lim_{i\to\infty} |\nu_i(A) - \nu_{j_0}(A)| \le \varepsilon$. Since this holds for all $\varepsilon > 0$, it must be that $\nu(A) = \sum_k \nu(A_k)$, and thus ν is a complex (Borel) measure. Since the sets A_k and $\varepsilon > 0$ were arbitrary, it is seen from (0.2) that $\|\nu - \nu_i\| \to 0$, as $i \to \infty$.

Exercise 4. Here we have a measure μ of order m, and numbers $\beta > \alpha^m$ (which are greater than 1) and $c > 0$. We'll show that for all $x \in \operatorname{spt}\mu$ there exists an (α, β)-doubling cube Q centred at x with side length $l(Q) \ge c$. Actually, let us with the same effort show that this holds for any $x \in \mathbb{R}^n$.

Fix some point x, and consider the cubes $Q(x, \alpha^k c), k = 0, 1, 2, \dots$, centred at x with side length $\alpha^k c$. The idea is that if for some k the cube $Q(x, \alpha^k c)$ is not doubling, then $\mu(Q(x, \alpha^{k+1} c)) > \beta\mu(Q(x, \alpha^k))$, and if this happens for all k then it implies that the measure of $Q(x, \alpha^k c)$ grows too fast for the measure μ to be of order m.

So to get a contradiction assume that none of the cubes $Q(x, \alpha^k c), k = 0, 1, 2, \dots$, is doubling. Then

$$\frac{\mu(Q(x, \alpha^{k+1} c))}{(\alpha^{k+1} c)^m} \ge \Big(\frac{\beta}{\alpha^m}\Big)^{k+1} \frac{\mu(Q(x, c))}{c^m} \to \infty, \ \ k \to \infty, \tag{0.4}$$

because $\beta > \alpha^m$. This is impossible since if μ is of order m the left hand side of (0.4) stays bounded.[2]

Exercise 5. Suppose again we have a measure μ of order m. Define the conical square function for any $f \in \bigcup_{p\in[1,\infty]} L^p(\mu)$ by

$$S_\mu f(x) := \Big(\iint_{\Gamma(x)} |\theta_t^\mu f(y)|^2 \frac{dt\, d\mu(y)}{t^{m+1}} \Big)^{\frac{1}{2}},$$

where $\Gamma(x) := \{(y, t) \in \mathbb{R}^{n+1}_+ : |y - x| < t\}$. We want to find some m-LP-family of kernels and a related vertical square function V_μ so that $\|S_\mu f\|_{L^2(\mu)} = \|V_\mu f\|_{L^2(\mu)}$ for all $f \in L^2(\mu)$.

[1] Note that in the definition of a complex measure we require $\nu(A) = \sum_k \nu(A_k)$ for sets as above. This means that the series should be convergent to $\nu(A)$ in *any* order, and this implies that the series is actually absolutely convergent. Conversely, if a series $\sum_{k=1}^{\infty} |a_k|$ of some complex numbers a_k converges absolutely, then $\sum_k a_k$ converges to the same value in any ordering of the numbers a_k.

[2] The definition of μ being of order m is stated in terms of balls, but it is easy to see that it can equivalently be done with cubes with a possibly different constant.

Let us just look what the $L^2(\mu)$-norm of the conical square function applied to some $f \in L^2(\mu)$ looks like. Using Fubini we have

$$\begin{aligned}
\|S_\mu f\|_{L^2(\mu)}^2 &= \int_{\mathbb{R}^n} \iint_{\mathbb{R}^{n+1}} 1_{\Gamma(x)}(y,t)|\theta_t^\mu f(y)|^2 \frac{dt\, d\mu(y)}{t^{m+1}} d\mu(x) \\
&= \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} 1_{\Gamma(x)}(y,t) d\mu(x) |\theta_t^\mu f(y)|^2 \frac{dt\, d\mu(y)}{t^{m+1}} \\
&= \int_{\mathbb{R}^n} \int_0^\infty \frac{\mu(B(y,t))}{t^m} |\theta_t^\mu f(y)|^2 \frac{dt}{t} d\mu(y).
\end{aligned}$$

So if we define the kernels for all $t > 0$ by

$$\mathbb{R}^n \times \mathbb{R}^n \ni (x,y) \mapsto \tilde{s}_t(x,y) := s_t(x,y) \Big(\frac{\mu(B(x,t))}{t^m} \Big)^{\frac{1}{2}},$$

then we see that in principle the vertical square function V_μ defined using these kernels satisfies $\|S_\mu f\|_{L^2(\mu)} = \|V_\mu f\|_{L^2(\mu)}$.

We still have to verify that $(\tilde{s}_t)_{t>0}$ is an admissible family of kernels, that is, it satisfies the required size- and y-Hölder estimates. But this is quite immediate since μ is of order m and thus $\frac{\mu(B(x,t))}{t^m} \lesssim 1$ for all x and t. Note that the family $(\tilde{s}_t)_{t>0}$ is not required to have any x-continuity properties.

Exercise 6. Again μ is a measure of order m. We want to show that there exists some constant $C > 0$ such that for an arbitrary ball B centred at c with radius r we have

$$|S_\mu(1_{\mathbb{R}^n \setminus 10B})(x) - S_\mu(1_{\mathbb{R}^n \setminus 10B})(c)| \le C$$

for all $x \in B$.

For any $u > 0$ define the truncate cones $\Gamma_u(x) := \{(y,t) \in \Gamma(x) : t > u\}$ and $\Gamma^u(x) := \{(y,t) \in \Gamma(x) : t \le u\}$, and also the corresponding truncated conical square functions $S_{\mu,u}$ and S_μ^u where the integration is carried over only the truncated cones. The conical square function of a function f at a point x can be thought of as the $L^2(\mathbb{R}^{n+1}_+, \frac{dtd\mu}{t^{m+1}})$-norm of the function $1_{\Gamma(x)}(y,t)\theta_t^\mu f(y)$. Using this point of view we have, for any $s > 0$, that

(0.5)
$$\begin{aligned}
&|S_\mu(1_{\mathbb{R}^n \setminus 10B})(x) - S_\mu(1_{\mathbb{R}^n \setminus 10B})(c)| \\
&\le \Big(\iint_{R^{n+1}_+} |1_{\Gamma(x)}(y,t)\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(y) - 1_{\Gamma(c)}(y,t)\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(y)|^2 \frac{dtd\mu(y)}{t^{m+1}} \Big)^{\frac{1}{2}} \\
&\le \Big(\iint_{R^{n+1}_+} |1_{\Gamma_u(x)}(y,t)\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(y) - 1_{\Gamma_u(c)}(y,t)\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(y)|^2 \frac{dtd\mu(y)}{t^{m+1}} \Big)^{\frac{1}{2}} \\
&+ S_\mu^u(1_{\mathbb{R}^n \setminus 10B})(x) + S_\mu^u(1_{\mathbb{R}^n \setminus 10B})(c).
\end{aligned}$$

(Here the first inequality was that in a normed space we have $\big|\|x\| - \|y\|\big| \le \|x - y\|$ and the last was just triangle inequality.) The point of this splitting is that for "small" t we can use the hole in the support of the function, and for "big" t the two shifted cones suitably cancel each other.

Indeed, suppose $(y,t) \in \Gamma(x)$. Then

$$
\begin{aligned}
|\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(y)| &\lesssim \int_{\mathbb{R}^n \setminus 10B} \frac{t^\alpha}{(t+|y-z|)^{m+\alpha}} d\mu(z) \\
&\lesssim \int_{\mathbb{R}^n \setminus 10B} \frac{t^\alpha}{(r+|y-z|)^{m+\alpha}} d\mu(z) \lesssim \frac{t^\alpha}{r^\alpha},
\end{aligned}
$$

where we estimated the integral as in (0.1). We can use this for, say, $t \le 10r$, as

$$
\begin{aligned}
S_\mu^{10r}(1_{\mathbb{R}^n \setminus 10B})(x)^2 &\lesssim \frac{1}{r^{2\alpha}} \iint_{\Gamma^{10r}(x)} t^{2\alpha} \frac{dt\, d\mu(y)}{t^{m+1}} \\
&= \frac{1}{r^{2\alpha}} \int_0^{10r} t^{2\alpha-1} \frac{\mu(B(x,t))}{t^m} dt \lesssim \frac{(10r)^{2\alpha}}{r^{2\alpha}} \lesssim 1,
\end{aligned}
$$

because μ is of order m. So if we choose for example $u = 10r$ in (0.5), then we see that the last two terms there to be estimated are in control.

Looking at the first term in the right hand side of (0.5) which we still need to estimate, we see that we have to integrate the function $\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(y)$ over the set

$$
\begin{aligned}
&(\Gamma_{10r}(x) \setminus \Gamma_{10r}(c)) \cup (\Gamma_{10r}(c) \setminus \Gamma_{10r}(x)) \\
&\subset \{(y,t) \in \mathbb{R}^{n+1} : |y-c| \ge 9r, t \in [|y-c|-r, |y-c|+r]\} =: A.
\end{aligned}
$$

Combining this with the fact that $|\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(y)| \lesssim 1$ for all (y,t) by (0.1), we get

$$
\begin{aligned}
&\iint_{R_+^{n+1}} |1_{\Gamma_s(x)}(y,t)\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(y) - 1_{\Gamma_s(c)}(y,t)\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(y)|^2 \frac{dt d\mu(y)}{t^{m+1}} \\
&\lesssim \iint_A \frac{dt d\mu(y)}{t^{m+1}} = \int_{\mathbb{R}^n \setminus B(c,9r)} \int_{|y-c|-r}^{|y-c|+r} \frac{dt}{t^{m+1}} d\mu(y) \\
&\sim \int_{\mathbb{R}^n \setminus B(c,9r)} \frac{r}{|y-c|^{m+1}} d\mu(y) \lesssim \frac{r}{9r} \lesssim 1,
\end{aligned}
$$

where a computation as in (0.1) was again used to estimate the integral. This completes the proof for the conical square function.

Suppose then we have a vertical square function V_μ with a kernel $(s_t)_{t>0}$ that satisfies also the x-Hölder assumption. Similarly as with the conical square function we have

$$
\begin{aligned}
&|V_\mu(1_{\mathbb{R}^n \setminus 10B})(x) - V_\mu(1_{\mathbb{R}^n \setminus 10B})(c)| \\
&\le \Big(\int_{2r}^\infty |\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(x) - \theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(c)|^2 \frac{dt}{t} \Big)^{\frac12} \\
&+ \Big(\int_0^{2r} |\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(x)|^2 \frac{dt}{t} \Big)^{\frac12} + \Big(\int_0^{2r} |\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(c)|^2 \frac{dt}{t} \Big)^{\frac12}.
\end{aligned}
$$

Using again $|\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(x)| \lesssim \frac{t^\alpha}{r^\alpha}$ for all $x \in B$ and $t > 0$, we get

$$
\begin{aligned}
&\Big(\int_0^{2r} |\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(x)|^2 \frac{dt}{t} \Big)^{\frac12} + \Big(\int_0^{2r} |\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(c)|^2 \frac{dt}{t} \Big)^{\frac12} \\
&\lesssim \Big(\int_0^{2r} \frac{t^{2\alpha}}{r^{2\alpha}} \frac{dt}{t} \Big)^{\frac12} \lesssim 1.
\end{aligned}
$$

Finally, the x-continuity of the kernel gives for all $t \geq 2r$ that

$$\begin{aligned}
&|\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(x) - \theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(c)| \\
&= \Big| \int_{\mathbb{R}^n} (s_t(x,y) - s_t(c,y)) 1_{\mathbb{R}^n \setminus 10B}(y) d\mu(y) \Big| \\
&\lesssim \int_{\mathbb{R}^n \setminus 10B} \frac{|x-c|^\alpha}{(t+|y-c|)^{m+\alpha}} d\mu(y) \lesssim \frac{r^\alpha}{t^\alpha},
\end{aligned}$$

and thus

$$\begin{aligned}
&\Big(\int_{2r}^\infty |\theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(x) - \theta_t^\mu(1_{\mathbb{R}^n \setminus 10B})(c)|^2 \frac{dt}{t} \Big)^{\frac{1}{2}} \\
&\lesssim \Big(\int_{2r}^\infty \frac{r^{2\alpha}}{t^{2\alpha}} \frac{dt}{t} \Big)^{\frac{1}{2}} \lesssim 1.
\end{aligned}$$

This concludes the estimate for the vertical square function.