Scaling limit of a sequence of functions

Suppose that for each value of the *scaling parameter* $L \in \mathbb{N}$ there is given a function $f_L(x) \in \mathbb{R},\ x \in \mathbb{R}$.

Do the functions have a *scaling limit*?

. . . that is, a function F such that

$$F(\xi) = \lim_{L\to\infty} f_L(\xi L), \quad \xi \in \mathbb{R}$$

Trivial example:

Choose $F : \mathbb{R} \to \mathbb{R}$ and define $f_L(x) := F(x/L)$

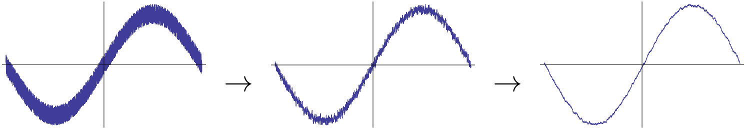
Why bother?

$$F(\xi) = \lim_{L\to\infty} f_L(\xi L), \quad \xi \in \mathbb{R}$$

If all f_L are known, why bother looking at the limit F?

Typically used when *f_L are not known explicitly* but the limit F can be computed independently, for instance, by solving a differential equation.

Example:



$f_1(\xi)$ $\quad$ $f_4(4\xi)$ $\quad$ $f_L(L\xi)$

Convergence in which sense?

$$F(\xi) = \lim_{L\to\infty} f_L(\xi L), \quad \xi \in \mathbb{R}$$

The goal is to *"retain only relevant degrees of freedom"*.

$\Rightarrow$ The choice of the function topology for the limit can be crucial.

Some standard alternatives:

1. *Pointwise convergence:* bad, does not preserve local densities.
   $f_L(x) := L\mathbb{1}(0<x<1) \Rightarrow f_L(\xi L) = L\mathbb{1}(0<\xi<L^{-1}) \to 0$
2. *Uniform convergence:* usually too strong. If all f_L are continuous and decay at infinity, then so must the limit F.
3. *L^1_{loc}-convergence:* better, preserves local densities and allows forgetting about troublesome things (like boundary conditions) "at infinity". However, still cannot handle "concentration of mass": $f_L(x) := L\mathbb{1}(0<x<1)$ has no limit.

Convergence as distributions

The scaling limit as a distribution: For all testfunctions φ require

$$\langle F, \varphi \rangle = \lim_{L \to \infty} \langle f_L, \varphi_L \rangle, \quad \text{where} \quad \varphi_L(x) := \frac{1}{L}\varphi(x/L)$$

- If $F \in L^1_{\text{loc}}$, we define here $\langle F, \varphi \rangle := \int_{\mathbb{R}} d\xi \, F(\xi)\varphi(\xi)$. Then any scaling limit F of f_L in L^1_{loc} is a scaling limit as a distribution.
- Using testfunctions which are close to normalized characteristic functions, we thus have for $\xi_0 \in \mathbb{R}$, $\delta > 0$,

$$\frac{1}{2\delta} \int_{|\xi - \xi_0| < \delta} d\xi \, F(\xi) \approx \frac{1}{2\delta L} \int_{|\xi - \xi_0 L| < \delta L} dx \, f_L(x)$$

  $\Rightarrow$ “local averages at the scale L” eventually coincide.
- In this way, concentration of mass is resolved as convergence to a Dirac δ-measure: $f_L(x) := L\mathbb{1}(0<x<1) \;\to\; \delta(x)$

An example from non-equilibrium statistical mechanics

How a macroscopic dynamical system of "spatial size" $L \gg 1$ with *normal thermal conductivity* is supposed to behave:

$t = 0$	"Typical" *initial state* (deterministic or stochastic)
$\downarrow$	$\downarrow$ (thermalization)
$t = O(L^\varepsilon)$	*Local equilibrium state* (stochastic): local statistics given by a thermal state (labeled by $T(x, t)$, ...)
$\downarrow$	$\downarrow$ (hydrodynamics)
$t = O(L^2)$	*Relaxation* towards equilibrium, hydrodynamics / Fourier's law