Department of Mathematics and Statistics
Fourier analysis
Exercise 4
Oct. 5, 2015

1. Suppose f is the 2π-periodic function given by $f(x) = \mathrm{sgn}(x) = x/|x|$ when $x \in (-\pi, \pi) \setminus \{0\}$, and $f(0) = f(\pm\pi) = 0$. Determine the Fourier series of f and show that for $N \geq 1$ an odd integer, you can write the partial series in the form

$$S_N f(x) = \frac{4}{\pi} \sum_{k=0}^{(N-1)/2} \frac{\sin((2k+1)x)}{2k+1}.$$

2. Suppose the sequence $(x_n)_{n=1}^{\infty}$ is equidistributed (mod 1) and $a \in \mathbb{Z} \setminus \{0\}$. Show that then also the sequence $(ax_n)_{n=1}^{\infty}$ is equidistributed (mod 1).
Does the result hold when $\alpha \notin \mathbb{Q}$?

3. Use the Dirichlet kernel $D_N(x) = \frac{\sin[(N+\frac{1}{2})\,x]}{\sin\frac{x}{2}}$ and the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$ to show that

$$\int_0^{\infty} \frac{\sin\, x}{x}\, dx := \lim_{M\to\infty} \int_0^M \frac{\sin\, x}{x}\, dx = \frac{\pi}{2}.$$

[Hint: Show that the function $g(x) = \frac{1}{\sin\frac{x}{2}} - \frac{2}{x}$ is continuous on the interval $[-\pi, \pi]$, and use the Riemann-Lebesgue Lemma.]

4. Show that the sequence $(< a\log n >)_{n=1}^{\infty}$ is *not* equidistributed (mod 1) for any $a \in \mathbb{R}$.

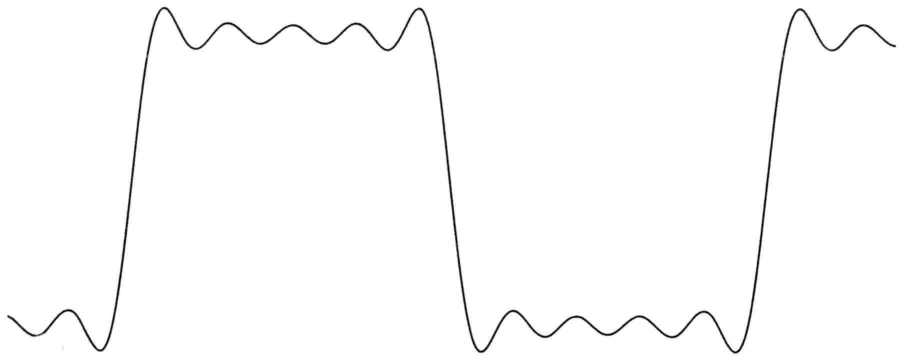
[Hint: Apply Weyl's criterium and compare the sum $\sum_{n=1}^{N} e^{2\pi i a \log n}$ with the corresponding integral.]

5. Suppose f is the function of Problem 1. Show that

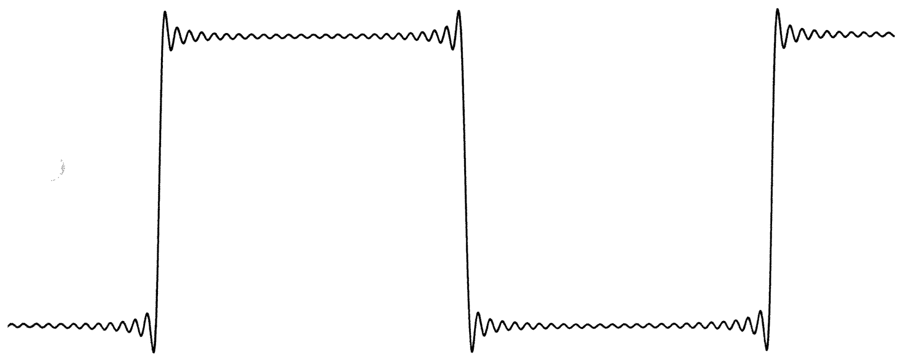
$$\lim_{N\to\infty} S_N f(\frac{\pi}{N}) = \frac{2}{\pi}\int_0^{\pi} \frac{\sin(x)}{x} dx =: G_0$$

and that $G_0 > 1$. This proves the *Gibbs phenomenon* for the function f, c.f. the pictures on the next page.

[Hint: In " $>$ " make use Problem 3.]



$S_N f(x)$, $N=5$



$S_N f(x)$, $N=25$