Stochastic analysis, fall 2014, Exercises-8, 26.11.2013

It is not true that all uniformly integrable local martingales are true martingales. Even local martingales bounded in $L^2(P)$ need not to be true martingales! Here we study such counterexample.

Let $\mathbf{B}_t = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)})$ a 3-dimensional brownian motion starting from 0 at time 0 , with independent components, so that $\langle B^{(i)}, B^{(j)}\rangle_t = \delta_{ij}$. The process

$$R_t = |B_t| = \sqrt{\sum_{i=1}^{3} \left(B_s^{(i)}\right)^2}$$

is called the 3-dimensional Bessel process.

1. Use Ito formula to compute the semimartingale decomposition of R_t into a continuous local martingale part W_t and a continuous process of finite variation.

   Solution

   Let $f(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then by Ito formula

$$\begin{aligned} f(\mathbf{B}_t) &= f(\mathbf{B}_0) + \sum_{i=1}^{3} \int_0^t \partial_i f(\mathbf{B}_s) dB^{(i)}(s) + \frac{1}{2} \sum_{i,j}^{3} \int_0^t \partial_i \partial_j f(\mathbf{B}_s) d\langle B^{(i)}, B^{(j)} \rangle_s \\ &= \sum_{i=1}^{3} \int_0^t \partial_i f(\mathbf{B}_s) dB^{(i)}(s) + \frac{1}{2} \sum_{i}^{3} \int_0^t \partial_i^2 f(\mathbf{B}_s) d\langle B^{(i)} \rangle_s \\ &= \sum_{i=1}^{3} \int_0^t \frac{B_s^{(i)}}{\sqrt{(B_s^{(i)})^2 + (B_s^{(i)})^2 + (B_s^{(i)})^2}} dB^{(i)}(s) + \int_0^t \frac{1}{\sqrt{(B_s^{(i)})^2 + (B_s^{(i)})^2 + (B_s^{(i)})^2}} ds \end{aligned}$$

   We see that the first term is an Ito integral, so it is a continuous local martingale and the second term is a continuous non decreasing process with finite variation. This means that R is a semimartingale. Note that the second term is P a.s. finite because

$$E\left(\int_0^t \frac{1}{\sqrt{(B_s^{(i)})^2 + (B_s^{(i)})^2 + (B_s^{(i)})^2}} ds \right) = E\left(\frac{1}{\sqrt{(B_1^{(i)})^2 + (B_1^{(i)})^2 + (B_1^{(i)})^2}} \right) \int_0^t \frac{ds}{\sqrt{s}} < \infty$$

   where we used the fact that $B_s^{(i)} \sim \sqrt{t} B_1$. This implies that

$$P\left(\int_0^t \frac{ds}{R_s} < \infty \right) = 1$$

2. Compute $\langle R \rangle_t = \langle W \rangle_t$ and use Paul Lévy's characterization theorem for Brownian motion to show that the local martingale part of R_t which satisifies

$$W_t = R_t - \int_0^t \frac{1}{R_s} ds$$

   is a Brownian motion in the filtration $\mathbb{F}$ generated by (B_t).

   Solution

   By proposition 27 we have

$$\langle W_t \rangle = \sum_{i=1}^{3} \int_0^t \frac{(B_s^{(i)^2})}{(B_s^{(i)})^2 + (B_s^{(i)})^2 + (B_s^{(i)})^2} d\langle B^{(i)} \rangle_s = \int_0^t ds = t$$

By the Doob-Meyer decompostion we have

$$W_t^2 = N_t + \langle W \rangle = N_t + t$$

where N_t is a local martingale, so that $W_t^2 - t$ is a local martingale and the Lev characterization theorem says that W_t is a Brownian Motion.

3. Show that R_t is a $\mathbb{F}$-submartingale.

   Solution
   By Jensen inequality one has

$$E(R_t|\mathcal{F}_s) = E(|\mathbf{B}_t||\mathcal{F}_s) \geq |E(\mathbf{B}_t|\mathcal{F}_s)| = |\mathbf{B}_s| = R_s$$

4. Let $M_t = R_t^{-1}$ for $t \geq 1$. We start the process at time 1 since $R_0 = 0$.
   Use Ito formula to show that $(M_t)_{t\geq 1}$ is a local martingale, and write its Ito integral representation.

   Solution
   Let $M_t = f(R_t) = R_t^{-1}$, then

$$\begin{aligned} M_t &= f(R_1) - \int_1^t \frac{dR_s}{R_s^2} + \int_1^t \frac{ds}{R_s^3} \\ &= f(R_1) - \int_1^t \frac{dW_s}{R_s^2} - \int_1^t \frac{ds}{R_s^3} + \int_1^t \frac{ds}{R_s^3} = \frac{1}{R_1} - \int_1^t \frac{dW_s}{R_s^2} \end{aligned}$$

   where we used exercise 2 to get $dR_s = dW_s - ds/R_s$. We see that M_t is a martingale by proposition 27.

5. Compute $\langle M \rangle_t$.

   Solution
   Again by proposition 27 we have

$$\langle M \rangle_t = \int_1^t \frac{d\langle W \rangle_s}{R_s^4} = \int_1^t \frac{ds}{R_s^4}$$

6. Show that (M_t) is a supermartingale. Hint: it is non-negative, you can use Fatou lemma for the sequence of localized martingales $(M_{t\wedge\tau_n} : t \geq 1)$, $n \in \mathbb{N}$.

   Solution
   Let τ_n be a localizing sequence for the martingale M_t. Since M_t is non negative, Fatou lemma gives

$$E(M_t|\mathcal{F}_s) = E(\lim_{n\to\infty} M_{t\wedge\tau_n}|\mathcal{F}_s) \leq \liminf_{n\to\infty} E(M_{t\wedge\tau_n}|\mathcal{F}_s) = \liminf_{n\to\infty} M_{s\wedge\tau_n} = M_s$$

7. Let $\tau_a := \inf\{t \geq 1 : R_t = a\}$, $a > 0$, with the convention $\inf\{\emptyset\} = \infty$.
   Show that the stopped process $(M_t^{\tau_a})_{t\geq 1}$ is a martingale and consequently $(\tau_{1/n} : n \in \mathbb{N})$ is a localizing sequence for the local martingale $(M_t : t \geq 1)$

   Solution
   Since $M_t^{\tau_a} \leq 1/a$, then by bounded convergence we have

$$E(M_t^{\tau_a}|\mathcal{F}_s) = \lim_{n\to\infty} E(M_{t\wedge\tau_n}^{\tau_a}|\mathcal{F}_s) = \lim_{n\to\infty} M_{s\wedge\tau_n}^{\tau_a} = M_s^{\tau_a}$$

   for a localizing sequence τ_n.

8. Let $0 < r' < y < r''$. Use the martingale property of $(M_{t\wedge\tau_r} : t \geq 1)$ to compute $P(\tau_{r'} < \tau_{r''}|R_1 = y)$. By the conditioning we mean that we start R_t at time $t = 1$ in position y.

   Solution
   Note that $M_{t\wedge\tau_{r'}}$ is a bounded martingale, so it is uniformly integrable and by Doob optional stopping time theorem we have
   $$E(M_{\tau_{r'}\wedge\tau_{r''}}) = \frac{1}{R_1}$$
   Moreover, we note that a priori $M_{\tau_{r'}\wedge\tau_{r''}}$ can assume three values: $1/r'$ when $\tau_{r'} < \tau_{r''}$, $1/r''$ when $\tau_{r'} > \tau_{r''}$ and some unknown value M_∞ when $\tau_{r'} = \tau'_{r'} = \infty$. We will show that actually $P(\tau_{r'} = \tau'_{r'} = \infty) = 0$.
   Infact, let us be $\tau_{r'} = \tau$ or $\tau_{r''} = \tau$. From Doob-Meier decomposition we know that
   $$M_t^2 = N_t + \langle M_t\rangle = N_t + \int_1^t \frac{ds}{R_s^4}$$
   where N_t is a local martingale, so that the stopped process $N_{t\wedge\tau}$ is a martingale bounded from above because
   $$N_{t\wedge\tau} = M_{t\wedge\tau}^2 - \int_0^{\tau\wedge t} \frac{ds}{R_s^4} \leq M_{t\wedge\tau}^2 \leq M_\tau^2$$
   By Doob convergence theorem, when we take the limit as $t \to \infty$ we get
   $$\frac{\tau - 1}{(r'')^4} \leq \int_1^\tau \frac{ds}{R_s^4} = N_\tau - M_\tau^2 \in L^1$$
   Then we get that $\tau \in L^1$ and since τ is non negative, we obtain that $P(\tau = \infty) = 0$. So we have
   $$\frac{1}{y} \quad = \quad E(M_{\tau_{r'}\wedge\tau_{r''}}) = P(\tau_{r'} < \tau_{r''}|R_1 = y)\frac{1}{r'} + (1 - P(\tau_{r'} < \tau_{r''}|R_1 = y))\frac{1}{r''}$$
   from which we get
   $$P(\tau_{r'} < \tau_{r''}|R_1 = y) = \frac{1/y - 1/r''}{1/r' - 1/r''}$$

9. For $0 < r < y$ compute also $P(\tau_r < \infty|R_1 = y)$.

   Solution
   Note that $\{\omega : \tau_r(\omega) < \infty\} = \{\omega : \cup_n^\infty \tau_r(\omega) < \tau_n\}$ where the sequence of events $\{\omega : \tau_r(\omega) < \tau_n\}$ is increasing because $\tau_n \to \infty$ as $n \to \infty$. Then we get
   $$P(\tau_r < \infty|R_1 = y) = P(\cup_{n\geq r}^\infty(\tau_r(\omega) < \tau_n)|R_1 = y) = \lim_{n\to\infty} P(\tau_r(\omega) < \tau_n|R_1 = y) = r/y$$
   Since
   $$P(\tau_r < \infty|R_1 = y) \to 0 \text{ as } r \downarrow 0$$
   by Borel Cantelli there is a sequence $r_n \downarrow 0$ with $\tau_{r_n}(\omega) = \infty$ for r_n small enough.
   This shows that $(\tau_{1/n}(\omega) : n \in \mathbb{N})$ is a localizing sequence for the local martingales $(M_t(\omega) : t \geq 1)$

10. Show that the 3-dimensional Brownian motion is transient, $|B_t| \to \infty$ P a.s., meaning that it leaves eventually any ball centered around the origin without coming back, and therefore $M_\infty = \lim_{t\to\infty} M_t = 0$.

Solution
Starting at $R_1 = y$, we consider the stopping time $\tau_{y/2}$. By 9, there is a probability of 1/2 that $R_t \to \infty$. Otherwise we hit $y/2$ and consider $\tau_{y/4}$. In this case we have a probability 1/4 that $R_t \to \infty$. By iterating this procedure, we get that

$$P(R_t \to \infty) = \sum_{j=1}^{\infty} 2^{-j} = 1$$

Alternatively, let

$$A_n = \{\omega : |B_t(\omega)| > n \forall t \geq \tau_{n^3}(\omega)\}$$

$P(A_n^c)$ is given by exercise 8 with $r' = n$ and $y = n^3$.

$$P(A_n^c) = \frac{n}{n^3} = n^{-2}$$

By Borel Cantelli lemma $P(\limsup A_n^c) = 0$, equivalently

$$P(\liminf A_n) = 1$$

Note also that $P(\tau_{n^3} < \infty) = 1$, since for the 3-dimensional Brownian motion $|B_t|^2 = |B_t^{(1)}|^2 + |B_t^{(2)}|^2 + |B_t^{(3)}|^2 \geq |B_t^{(i)}|^2 \to \infty$ as $t \to \infty$. and we have seen that the one dimensional Brownian motion reaches any level with probability 1.

11. Using the multivariate gaussian density in polar coordinates, compute the probability densities of R_t and M_t, and show that the local martingale $(M_t : t \geq 1)$ is bounded in L^2 , so that in particular it is uniformly integrable. (We start the martingale at time 1 since there is the square norm explodes near the origin).
Note however that $(M_t)_{t \geq 1}$ is not a martingale. Otherwise (X_t) would be an uniformly integrable martingale so that $M_t = E(M_\infty | \mathcal{F}_t)$, $t \geq 1$. But in dimension 3 the Brownian motion is transient, which means that $M_\infty = 0$.

Solution
The multivariate Gaussian density is given by

$$P(\mathbf{B}_t \in d^3x) = \frac{1}{(2\pi t)^{3/2}} e^{-|x|^2/2t} d^3x$$

Passing to polar coordinate we have $d^3x = \rho^2 \sin\theta d\theta d\phi d\rho$. To get the marginal distribution of the radius varaible ρ we just need to intergate over θ and ϕ:

$$P(R_t \in d^3x) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \frac{1}{(2\pi t)^{3/2}} e^{-\rho^2/2t} \rho^2 \sin\theta d\rho = \sqrt{\frac{2}{\pi t^3}} e^{-\rho^2/2t} \rho^2 d\rho$$

Therefore

$$P(M_t \in du) = \frac{d}{du} P(M_t \leq u) = \frac{d}{du} P(R_t \geq 1/u) = \sqrt{\frac{2}{\pi t^3}} e^{-u^{-2}/2t} u^{-4} du$$

Now we can estimate the L^2 norm of M_t:

$$E(M_t^2) = \sqrt{\frac{2}{\pi t^3}} \int_0^\infty u^2 e^{-u^{-2}/2t} u^{-4} du = \sqrt{\frac{2}{\pi t^3}} \int_0^\infty e^{-v^{-2}/2t} dv = \sqrt{\frac{2}{\pi t^3}} \frac{\sqrt{2\pi t}}{2} = \frac{1}{t} < \infty$$

12. Compute also the probability distribution of R_t^2.

Show first that for $t = 1$,

$$P(R_1^2 \in dx) = \mathbf{1}(x \geq 0) \frac{1}{\Gamma(3/2) 2^{3/2}} \exp(-x/2) x^{\frac{3}{2}-1} dx$$

which is the distribtion of a Gamma random variable with shape parameter 3/2 and scale parameter 2, (also called chi-square with 3 degrees of freedom and use the scaling property of Brownian motion.

Solution

By the previous exercise we know

$$P(R_t^2 \leq u) = P(R_t \leq \sqrt{u}) = \int_0^{\sqrt{u}} P(R_t \in dv)$$

so we get

$$P(R_t^2 \in du) = \frac{d}{du} \int_0^{\sqrt{u}} P(R_t \in dv) = \sqrt{\frac{2}{\pi t^3}} \frac{e^{-u/2t} u}{2\sqrt{u}} du = \frac{e^{-u/2t}\sqrt{u}}{\sqrt{2\pi t^3}} du$$

13. Show that $E(\langle M \rangle_t) = \infty \ \forall t \geq 1$.

Remark This is not in contradiction with $E(M_\infty^2) < \infty$, since

$$E\big((M_t - M_1)^2\big) = E\big(\langle M \rangle_t - \langle M \rangle_1\big)$$

holds for martingales but does not need to hold for local martingales. Even if the local martingale M_t is bounded in L^2, it means that M_t^2 is bounded in L^1 which does not give the uniform integrability condition which is necessary to take the limit of a localizing sequence under the expectation.

Solution

By exercise 5 we have

$$E(\langle M_t \rangle) = \int_1^t ds \int_0^\infty dx \frac{1}{x^2} \frac{e^{-x/2s}\sqrt{x}}{\sqrt{2\pi s^3}} = \infty$$

since $x^{-3/2}$ is not integrable in 0.

Remark In general, when M_t is a continuous local martingale with localizing sequence $\tau_n \uparrow \infty$, to show that it is a true martingale, you need to show that for $s \leq t$ and $A \in \mathcal{F}_s$

$$E_P(M_{t \wedge \tau_n} \mathbf{1}_A) \overset{?}{\to} E_P(M_t \mathbf{1}_A)$$
$$E_P(M_{s \wedge \tau_n} \mathbf{1}_A) \overset{?}{\to} E_P(M_s \mathbf{1}_A)$$

where the left sides are equal by since $(M_{t \wedge \tau_n} : t \geq 0)$ is a martingale. When the local martingale $(M_t : t \geq 0)$ is uniformly integrable or bounded in $L^2(P)$, for fixed t, the sequence $(M_{t \wedge \tau_n} : n \in \mathbb{N})$ does not need to be uniformly integrable, which is the condition that we need to take the limit inside the expectation.

Of course if the local martingale is bounded on finite intervals, $|M_t(\omega)| \leq c(t) < \infty$ P a.s., then it is a true martingale.

Also, when $M_t \in L^2(P)$ is a martingale and $\forall t$ and Y_t is a progressive integrand with

$$E\bigg(\int_0^t Y_s^2 d\langle M \rangle_s\bigg) < \infty \quad \forall t \tag{1}$$

then the Ito integral $(Y \cdot M)_t \in L^2(P)$ is a true martingale.

If M_t is not a square integrable martingale, or Y does not satisfy the condition 1, the stochastic integral $(Y \cdot M)_t$ is just local martingale.