Stochastic analysis, Fall 2014, Exercises-7, 12.07.2014

1. Let $\tau(\omega) \in [0, +\infty]$ be a random time, $F(t) = P(\tau \le t)$ for $t \in [0, \infty)$.
Consider the single jump *counting process* $N_t(\omega) = \mathbf{1}(\tau(\omega) \le t)$ which generates the filtration $\mathbb{F} = (\mathcal{F}_t^N)$ with $\mathcal{F}_t^N = \sigma(N_s : s \le t)$.

   (a) Show that τ is a stopping time in the filtration $\mathbb{F}$.
   Solution
   Obviously

$$\{\omega : \tau(\omega) \le t\} \in \sigma(\{\tau(\omega) \le s\} : s \le t) = \mathcal{F}_t^N$$

   (b) Show first that for every Borel function $f(x)$, the random variable

$$f(\tau(\omega))\mathbf{1}(\tau(\omega) \le s)$$

   is $\mathcal{F}_s$-measurable.
   Solution :
   If $f(x) = \mathbf{1}(x \in (a, b])$, with $a < b$,

$$f(\tau(\omega))\mathbf{1}(\tau(\omega) \le s) = \mathbf{1}(\tau \in (a \wedge s, b \wedge s]) = \mathbf{1}(\tau \le b \wedge s) - \mathbf{1}(\tau \le a \wedge s)$$

   which is $\mathcal{F}_s$-measurable.

   (c) Define the *cumulative hazard function*

$$\Lambda(t) = \int_0^t \frac{1}{1 - F(s-)} F(ds)$$

   where $F(s-) = P(\tau < s)$ denotes the limit from the left.
   Show that

$$M_t = N_t - \Lambda_{t \wedge \tau}$$

   is a an $\mathbb{F}$-martingale.
   Hint: use the definition, and show that for $s \le t$ and every $A \in \mathcal{F}_s$

$$E_P\Big((N_t - N_s)\mathbf{1}_A \Big) = E_P\Big((\Lambda_{t\wedge\tau} - \Lambda_{s\wedge\tau})\mathbf{1}_A \Big)$$

   It turns out that it is enough to do the computation for $A = \{\omega : \tau(\omega) > s\}$ (why ?). Fubini's theorem may be also useful.
   Solution
   It is enough to do the computation for A because for $\{\omega : \tau(\omega) \le s\}$ the relation is trivially satisfied.

$$E_P\big((N_t - N_s)\mathbf{1}(\tau > s)\big) = P(\tau \in (s, t]) = F(t) - F(s)$$

$$E_P\big((\Lambda_{\tau\wedge t} - \Lambda_{\tau\wedge s})\mathbf{1}(\tau > s)\big) = \int_0^\infty \big(\Lambda_{r\wedge t} - \Lambda_{r\wedge s}\big)\mathbf{1}(r > s)F(dr)$$

$$= \int_s^\infty \left(\int_{s\wedge r}^{t\wedge r} \frac{1}{1 - F(u-)} F(du) \right) F(dr) = \int_s^\infty \int_s^r \frac{1(u \le t)}{1 - F(u-)} F(du)F(dr)$$

$$= \int_s^\infty \left(\int_u^\infty F(dr) \right) \frac{1(u \le t)}{1 - F(u-)} F(du) = \int_s^t \frac{1 - F(u-)}{1 - F(u-)} F(du) = F(t) - F(s)$$

(d) Assume that $t \mapsto F(t)$ and therefore also $t \mapsto \Lambda(t)$ are continuous, which means $P(\tau = t) = 0\ \forall t \in \mathbb{R}^+$ and $P(\tau = \infty) = 0$.

Show that Λ_τ has 1-exponential distribution:

$$P\big(\Lambda_\tau > x) = \exp(-x), \quad x \geq 0$$

Hint: one line of proof compute the Laplace transform

$$\mathcal{L}(\theta) := E_P\bigg(\exp(-\theta\Lambda_\tau)\bigg) \quad \theta > 0$$

and compare it with the Laplace transform of the 1-exponential distribution.

Solution Note that when $t \mapsto F(t)$ is continuous, by the change of variables formula

$$d\log(1 - F(t)) = -\frac{1}{1 - F(t)}F(dt)$$

$$E_P\bigg(\exp(-\theta\Lambda_\tau)\bigg) = \int_0^\infty \exp\bigg(-\theta\int_0^t \frac{1}{1-F(s)}F(ds)\bigg)F(dt)$$
$$\int_0^\infty \exp\bigg(\theta\int_0^t d\log(1-F(s))\bigg)F(dt) = \int_0^\infty \exp\bigg(\theta(\log(1-F(t)) - \log(1))\bigg)F(dt) =$$
$$\int_0^\infty (1-F(t))^\theta F(dt) = \frac{1}{1+\theta} = \int_0^\infty \exp(-(1+\theta)t)dt$$

(e) Assuming again that $F(t)$ is continuous, show that the martingale M_t is uniformly integrable. What is M_∞?.

Note that $|M_t|$ is uniformly bounded in L^1, in fact

$$|M_t(\omega)| \leq \big(1 + \Lambda_{\tau(\omega)}\big) \in L^1(P), \qquad \forall t \geq 0$$

thus it is uniformly integrable. Note that both N_t and $\Lambda_{t\wedge\tau}$ are non-negative and non-decreasing processes, so that the limit is $M_\infty = 1 - \Lambda_\tau$.

2. Let $(M_t : t \in \mathbb{R}^+)$ a $\mathbb{F}$-martingale, and $\mathbb{G}$ a filtration with $\mathcal{G}_t \subseteq \mathcal{F}_t$ We assume that (M_t) is also $\mathbb{G}$-adapted. Show that (M_t) is a martingale in the smaller filtration $\mathbb{G}$.

   Solution
   $E(M_t|\mathcal{G}_s) = E(E(M_t|\mathcal{F}_s)|\mathcal{G}_s) = E(M_s|\mathcal{G}_s) = M_s$

3. Let $(M_t : t \in \mathbb{R})$ a F-martingale under P, and $\mathcal{G}_t$ a filtration such that $\forall t \geq 0$, the σ-algebrae $\mathcal{G}_t$ and $\sigma(M_s : s \leq t)$ are P-independent.

   Show that under P, $(M_t : t \in \mathbb{R}^+)$ is a martingale in the enlarged filtration $(\mathcal{F}_t \vee \mathcal{G}_t : t \geq 0)$.

   Solutions This is not always true. What is true is that $(M_t : t \in \mathbb{R}^+)$ is a martingale in the enlarged filtration $(\sigma(M_s : s \leq t) \vee \mathcal{G} : t \geq 0)$. when the σ-algebra $\mathcal{G}$ is P-independent from $(M_t : t \in \mathbb{R}^+)$

   If $G \in \mathcal{G}$ and $A \in \sigma(M_r : r \leq s)$ for $s \leq t$,

$$E_P((M_t - M_s)\mathbf{1}_{A\cap G}) = E_P((M_t - M_s)\mathbf{1}_A\mathbf{1}_G) = E_P((M_t - M_s)\mathbf{1}_A)P(G) = 0$$

   and the result follows since

$$\sigma(M_r : r \leq s) \vee \mathcal{G} = \sigma\big(A \cap G : A \in \sigma(M_r : r \leq s), G \in \mathcal{G}\big)$$

Counterexample

Let X_1, X_2, X_3 i.i.d. binary variables with $P(X_i = 1) = P(X_i = 0) = 1/2$, and $X_4 = (X_1 + X_2 + X_3) \mod 2$.

It follows that the distribution of (X_1, X_2, X_3, X_4) is invariant under permutations,

and for each distinct triple $1 \le i \ne j \ne k \le 4$ and $a, b, c \in \{0, 1\}$

$$P(X_i = a, X_j = b, X_k = c) = 2^{-3}$$

The random variables (X_1, X_2, X_3, X_4) are 3-wise independent but are not independent, since any three random variables determine the 4-th.

Let $M_0 = (X_3 - 1/2)$, $M_1 = (X_3 + X_4 - 1)$, $\mathcal{F}_0 = \sigma(X_2, X_3) \subseteq \mathcal{F}_1 = \sigma(X_2, X_3, X_4)$.

Now $(M_t : t = 0, 1)$ is a martingale in the filtration $(\mathcal{F}_t : t = 0, 1)$.

But M_t is not a martingale in the enlarged filtration $(\mathcal{F}_t \vee \sigma(X_1))$, because $M_1 \ne M_0$ are both $\sigma\mathcal{F}_0 \vee \sigma(X_1)$ measurable.

We construct a counterexmple there exist finitely exchangeale binary random variables (X_1, X_2, X_3, X_4) with values in $\{0, 1\}$

which are 3-wise independent but not fourwise independent, In other words words for every distinct $i \ne j \ne k$

$$P(X_i = a, X_j = b, X_k = c) = 2^{-3}, \forall a, b, c \in \{0, 1\}$$

but for some a, b, c, d

$$P(X_1 = a, X_2 = b, X_3 = c, X_d = d) \ne 2^{-4}$$

4. Let $(B_t : t \ge 0)$ a Brownian motion in the filtration $\mathbb{F}$, which means
   - $B_0(\omega) = 0$
   - $t \mapsto B_t(\omega)$ is continuous
   - $\forall 0 \le s \le t$, $(B_t - B_s)$ is P-independent from B_s, conditionally Gaussian with conditional mean $E(B_t - B_s|B_s) = 0$ and conditional variance $E((B_t - B_s)^2|B_s) = t - s$

   (a) Show that for $a > 0$ the process $(a^{-1/2}B_{at} : t \in \mathbb{R}^+)$ is also a Brownian motion.
   Solution
   Note that the last property of the Brownian motion is equivalent to require the increments $(B_t - B_s)$ to be independent from B_s and to be distributed as $G(0, t - s)$.
   Let be $W_t := a^{-1/2}B_{at}$ and note that W is a just a rescaled version of B. Then we have
   - $W_0 = 0$
   - W_t is continuous because B_t is continuous
   - $W_t - W_s = a^{-1/2}(B_{at} - B_{as}) \perp a^{-1/2}B_{as} = W_s$. Furthermore, W_t is Gaussian because B_t is, thus W_t is also conditionally Gaussian. Finally we check that $W_t - W_s$ has the expected mean and variance:

   $$\begin{aligned} E(W_t - W_s) &= a^{-1/2}E(B_{at} - B_{as}) = 0 \\ E((W_t - W_s)^2) &= a^{-1/2}E((B_{at} - B_{as})^2) = t - s \end{aligned}$$

   (b) The process $W_0 = 0$, $W_t = tB_{1/t}$ is also a Brownian motion.
   Solution

- $W_0 = 0$
- W_t is trivially continuous for $t > 0$. To show that it continuous in $t = 0$ we need to prove that almost surely
$$\lim_{t\to 0} W_t = 0$$
which is equivalent to show that
$$\lim_{t\to\infty} \frac{B_t}{t} = 0$$
which is a continuous version of the strong law of large numbers and it is equivalent to
$$\lim_{n\to\infty} \sup_{t>n} |B_t|/t = 0$$
P-a.s. We show that
$$\lim_{n\to\infty} \left(\frac{|B_n|}{n} + \frac{1}{n} \sup_{t\in(n,n+1]} |B_t - B_n| \right) = 0$$
Now $B_n/n \to 0$ P a.s. by the strong law of large numbers.
Also let $X_n = \sup_{t\in(n,n+1]} |B_t - B_n|$. Note that $(X_n : n \in \mathbb{N})$ are i.i.d., and by Doob submartigale inequality
$$P(X_1 > c) = P(\sup_{t\in(0,1]} B_t^2 > c^2) \le c^{-1} E(B_1^2) = c^{-2}$$
where the submartingale inequality holds in continuous time, because it holds in discrete time and by taking monotone limit it holds also when we take the supremum over the dyadics, and since the Brownian path is continuous it holds also in continuous time.
Therefore for all $c > 0$
$$\sum_n P(X_n > cn) \le c^{-2} \sum_n n^{-2} < \infty$$
by Borel Cantelli $P(\limsup_n \{X_n > cn\}) = 0$, which implies
$$P(\bigcap_m \bigcup_n \bigcap_{k>n} \{\frac{X_k}{k} > \frac{1}{m}\}) = 0 \Longleftrightarrow P\big(\lim_n \frac{X_n}{n} = 0\big) = 1$$
- $W_t - W_s =$ First, we check that $W(t) - W(s) \sim G(0, t-s)$ with $s \le t$ by looking at the characteristic function:
$$\begin{aligned} E\exp\{i\theta(W(t) - W(s))\} &= E\exp\left\{i\theta[s(B_{1/t} - B_{1/s}) + (t-s)B_{1/t}]\right\} = \\ &= E\exp\left\{i\theta s(B_{1/t} - B_{1/s})\right\} E\exp\left\{i\theta(t-s)B_{1/t}\right\} = \\ &= E\exp\left\{-i\theta s(B_{1/s} - B_{1/t})\right\} E\exp\left\{i\theta(t-s)W(1/t)\right\} = \\ &= \exp\left\{-\frac{1}{2}(\theta s)^2 \left(\frac{1}{s} - \frac{1}{t}\right)\right\} \exp\left\{-\frac{1}{2t}\theta^2(t-s)^2\right\} = \\ &= \exp\left\{-\frac{1}{2}\theta^2(t-s)\right\} \end{aligned} \tag{1}$$
Using the same decomposition as before, one can check that the characteristic function of $W_s + (W_t - W_s)$ factorizes, i.e
$$E\exp\{i\theta[(W(t) - W(s)) + W_s]\} = E\exp\{i\theta(W(t) - W(s))\} E\exp\{i\theta W_s\}$$

(c) Let $\theta \in \mathbb{R}$, and $i = \sqrt{-1}$ the imaginary unit
Show that

$$E_P\big(\exp(i\theta B_t)\big) = \exp\big(-\frac{1}{2}\theta^2 t\big)$$

Hint: Use complex integration over the rectangular contour delimited by in the complex plane by the points $R, (R+i\theta), (-R+i\theta), -R$ with $R \in \mathbb{R}$ and let $R \to \infty$.
Solution We can ignore the hint and just compute

$$E_P\big(\exp(i\theta B_t)\big) = \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t + i\theta x} = e^{-\frac{\theta^2 t}{2}} \int_{\mathbb{R}} dx \frac{1}{\sqrt{2\pi t}} e^{-(\frac{x}{\sqrt{2t}} - i\theta\sqrt{\frac{t}{2}})^2} = e^{-\frac{\theta^2 t}{2}}$$

(d) For $\theta \in \mathbb{R}$, consider now

$$M_t = \exp\big(i\theta B_t + \frac{1}{2}\theta^2 t\big) = \Big\{\exp\big(\frac{1}{2}\theta^2 t\big)\cos(\theta B_t) + \sqrt{-1}\exp\big(\frac{1}{2}\theta^2 t\big)\sin(\theta B_t)\Big\} \in \mathbb{C}$$

where $i = \sqrt{-1}$ is the imaginary unit.
Recall that $E(\exp(i\theta G)) = \exp(-\theta^2\sigma^2/2)$ when $G(\omega) \sim \mathcal{N}(0, \sigma^2)$.

- Show that M_t is complex valued $\mathbb{F}$-martingale, which means that real and imaginary parts are $\mathbb{F}$-martingales.
- Show that $\lim_{t\to\infty} |M_t(\omega)| = \infty$

Solution

- Note that M_t is integrable for any $t \in \mathbb{R}$ since $|M_t| \le e^{\theta^2 t/2}$. Moreover, the martingale property holds:

$$E(M_t|\mathcal{F}_s) = e^{\theta^2 t/2} E(e^{i\theta(B_t - B_s)} e^{i\theta B_s}|\mathcal{F}_s) = e^{\theta^2 t/2} e^{i\theta B_s} e^{\theta^2(t-s)/2} = M_s$$

- We have pointwise

$$\lim_{t\to\infty} |M_t| = \lim_{t\to\infty} e^{\theta^2 t/2} = \infty$$