Stochastic analysis, fall 2014, Exercises-6, 29.10.2014

Consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with the dicrete-time filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$

1. In discrete time, show that a $\mathbb{F}$-predictable $(P, \mathbb{F})$-martingale is constant, i,e $M_n(\omega) = M_0(\omega)$ $\forall n$.

   Solution

$$M_n = E(M_n | \mathcal{F}_{n-1}) = M_{n-1}$$

   then $M_n(\omega) = M_0(\omega)$ $\forall n$.

2. A *potential* $(Z_n : n \in \mathbb{N})$ is a non-negative $(P, \mathbb{F})$-supermartingale with

$$\lim_{n\to\infty} E(Z_n) = 0$$

   Show that a potential is uniformly integrable.

   Solution
   Note that, since Z_n is non-negative, then Z_n converges to 0 in L^1. Then, by theorem 12 in the lecture notes, we have that is uniformly integrable.

3. An $(\mathbb{F}, P)$-supermartingale $(X_n : n \in \mathbb{N})$ has *Riesz decomposition* if it can be written as

$$X_n = Y_n + Z_n$$

   where Y_n is a martingale and Z_n is a potential.

   (a) Show that if $\sup_{n\in\mathbb{N}} E_P\big(X_n^-\big) < \infty$ then X_n has Riesz decomposition with

$$Y_n = M_n - E(A_\infty|\mathcal{F}_n), \quad Z_n = E(A_\infty|\mathcal{F}_n) - A_n,$$

   where $X_n = M_n - A_n$ is the Doob decomposition of X into a martingale part M and a predictable part with A non-decreasing and $A_0 = 0$.

   (b) Show that the Riesz decomposition is unique.

   Solution

   (a) By the Doob convergence theorem and Doob decomposition we have that there exist almost surely

$$\lim_{n\to\infty} X_n = X_\infty = M_\infty - A_\infty \in L^1$$

   Furthermore, note that

$$E(A_n) = E(M_n) - E(X_n) \le E(M_0) + E(X_n^-)$$

   so we get

$$\sup_n E(A_n) < \infty$$

   By linearity we get

$$\begin{aligned} X_n &= M_n + E(M_\infty - A_\infty|\mathcal{F}_n) - E(M_\infty - A_\infty|\mathcal{F}_n) - A_n = \\ &= M_n - E(A_\infty|\mathcal{F}_n) + E(A_\infty|\mathcal{F}_n) - A_n = Y_n + Z_n \end{aligned}$$

   We check that Y_n is a martingale: it is integrable since M_n is a martingale

$$E(|M_n - E(A_\infty|\mathcal{F}_n)|) \le E|M_n| + E(E(A_\infty|\mathcal{F}_n)) = E|M_n| + E(A_\infty) < \infty$$

and by the tower property the martingale property holds

$$E(M_n - E(A_\infty|\mathcal{F}_n)|\mathcal{F}_{n-1}) = M_{n-1} - E(E(A_\infty|\mathcal{F}_n)|\mathcal{F}_{n-1}) = M_{n-1} - E(A_\infty|\mathcal{F}_{n-1}).$$

Now we check that Z_n is a potential: it is clearly non negative and it is integrable

$$E|Z_n| = E(E(A_\infty|\mathcal{F}_n) - A_n) \le E(E(A_\infty|\mathcal{F}_n)) = E(A_\infty) < \infty$$

and the super-martingale property holds

$$E(Z_n|\mathcal{F}_{n-1}) = E(E(A_\infty|\mathcal{F}_n) - A_n|\mathcal{F}_{n-1}) \le E(A_\infty|\mathcal{F}_{n-1}) - A_{n-1}$$

Finally we check that $\lim_{n\to\infty} E(Z_n) = 0$:

$$\lim_{n\to\infty} E(Z_n) = \lim_{n\to\infty} E(E(A_\infty|\mathcal{F}_n) - A_n) = E(A_\infty) - E(\lim_{n\to\infty} A_n) = 0$$

where we used monotone convergence in the last equality.

(b) Let be $X_n = Y'_n + Z'_n = Y_n + Z_n$, then we have

$$Y'_n - Y_n = Z_n - Z'_n$$

We know that both Z_n and Z'_n converge to 0 both in probability and in L^1 since they are uniformly integrable, then $(Y'_n - Y_n)$ converges to 0 both in probability and in L^1. But $(Y'_n - Y_n)$ is a martingale, so almost surely $\forall n$

$$(Z_n - Z'_n) = (Y'_n - Y_n) = E(Y'_\infty - Y_\infty|\mathcal{F}_n) = 0$$

4. Show that a martingale $(X_t : t \in \mathbb{N})$ has *Krickeberg decomposition*

$$X_t = L_t - M_t$$

where L_t and M_t are non-negative $(P, \mathbb{F})$-martingales, if and only if

$$\sup_{t\in\mathbb{N}} E_P\big(|X_t|\big) < \infty$$

Hints: You can always assume without loss of generality that $M_0 = 0$, otherwise consider the martingale $(M_t - M_0)$.
For sufficiency show take the decomposition $X_t = X_t^+ - X_t^-$, and show first that $(-X_t^-)$ is a a supermartingale and which admits a Riesz decomposition

$$(-X_t^-) = Y_t + Z_t$$

where Y_t is a martingale and Z_t is a potential. Show then that X_t has Krickeberg decomposition with

$$L_t = (X_t - Y_t) = X_t^+ + Z_t \ge 0, \quad \text{and} \quad M_t = -Y_t = X_t^- + Z_t \ge 0.$$

Solution
For sufficiency, we start by showing that $-X_t^-$ is a super-martingale: since the map $f : x \mapsto -x^-$ is concave, then the Jensen inequality for conditional expectation implies

$$E(f(X_t)|\mathcal{F}_{t-1}) \le f(E(X_t|\mathcal{F}_{t-1})) = -X_{t-1}^-$$

Then we get immediately that $-X_t^-$ has a Riesz decomposition $-X_t^- = Y_t + Z_t$, for the hypothesis $\sup_{t\in\mathbb{N}} E_P\big(|X_t|\big) < \infty$.
Now write

$$X_t = X_t - Y_t + Y_t$$

and define $L_t := X_t - Y_t$ and $M_t := -Y_t$, which are non-negative martingales.
To show the necessity we need to show that if $X_t = L_t - M_t$, where L_t, M_t are non-negative martingales, then

$$\sup_t E(|X_t|) < \infty$$

and this is true because $\forall t$

$$E(|X_t|) \le E(L_t) + E(M_t) = E(L_0) + E(M_0) < \infty$$

5. Suppose we have an urn which contains at time $t = 0$ two balls, one black and one white. At each time $t \in N$ we draw uniformly at random from the urn one ball, and we put it back together with a new ball of the same colour.

We introduce the random variables
$X_t(\omega) = \mathbf{1}\big\{$ the ball drawn at time t is black $\big\}$
and denote $S_t = (1 + X_1 + \cdots + X_t)$,
$M_t = S_t/(t+2)$, the proportion of black balls in the urn.
We use the filtration $\{\mathcal{F}_n\}$ with $\mathcal{F}_n = \sigma\{X_s :\ s \in \mathbb{N},\ s \le t\}$.

(a) Compute the Doob decomposition of (S_t), $S_t = S_0 + N_t + A_t$, where (N_t) is a martingale and (A_t) is predictable.

(b) Show that (M_t) is a martingale and find the representation of (M_t) as a martingale transform $M_t = (C \cdot N)_t$, where (N_t) is the martingale part of (S_t) and (C_t) is predictable.

(c) Note that the martingale $(M_t)_{t\ge 0}$ is uniformly integrable (Why ?). Show that P a.s. and in L^1 exists $M_\infty = \lim_{t\to\infty} M_t$. Compute $E(M_\infty)$.

(d) Show that $P(0 < M_\infty < 1) > 0$.

Since $M_\infty(\omega) \in [0,1]$, it is enough to show that $0 < E(M_\infty^2) < E(M_\infty)$ with strict inequalities.

Hint: compute the Doob decomposition of the submartingale (M_t^2), and than take expectations before going to the limit to find the value of $E(M_\infty^2)$.

Solution

(a) We recall that according to the Doob decomposition we have

$$N_t = \sum_{r=1}^{t} (S_r - E(S_r|\mathcal{F}_{r-1})) \quad A_t = \sum_{r=1}^{t} (E(S_r|\mathcal{F}_{r-1}) - S_{r-1})$$

where

$$E(S_t|\mathcal{F}_{t-1}) = S_{t-1} + E(X_t|\mathcal{F}_{t-1}) = S_{t-1} + M_{t-1} = S_{t-1}\left(1 + \frac{1}{t+1}\right)$$

So the Doob decomposition is

$$S_t = 1 + \sum_{r=1}^{t} S_{r-1}\frac{1}{t+1} + \sum_{r=1}^{t} (X_r - M_{r-1})$$

Since the predictable part is non-decreasing, we see that S_t is a submartingale.

(b) M_t is a martingale because

$$E\big(M_t|\mathcal{F}_{t-1}\big) = \frac{1}{t+2}E(S_t|\mathcal{F}_{t-1}) = \frac{1}{t+2}\bigg(1+\frac{1}{t+1}\bigg)S_{t-1} = \frac{S_{t-1}}{t+1} = M_{t-1}$$

and in order to find its representation as a martingale transform we consider

$$M_t - M_{t-1} = \frac{S_t}{t+2} - \frac{S_{t-1}}{t+1} = \frac{S_{t-1}+X_t}{t+2} - \frac{S_{t-1}}{t+1} = \frac{1}{t+2}\bigg(X_t + S_{t-1}\bigg(1-\frac{t+2}{t+1}\bigg)\bigg)$$
$$= \frac{1}{t+2}\bigg(X_t - \frac{S_{t-1}}{t+1}\bigg) = \frac{1}{t+2}(X_t - M_{t-1})$$

Therefore

$$M_t = \frac{1}{2} + \sum_{r=1}^{t}\frac{1}{r+2}(X_r - M_{r-1}) = \frac{1}{2} + (C\cdot N)_t$$

with $C_t = \frac{1}{t+2}$ (deterministic).

(c) M_t is uniformly integrable because

$$|M_t| = \frac{|S_t|}{t+2} \le \frac{t+1}{t+2} < 1$$

By Doobs' martingale convergence theorem $M_t(\omega) \to M_\infty(\omega)$ P-almost surely, and by uniform integrability also in $L^1(P)$. Since M_t is uniformly integrable, then $E(M_\infty) = E(M_t) = E(M_0) = 1/2$.

(d) First we show that if $0 < E(M_\infty^2) < E(M_\infty)$ then $P(0 < M_\infty < 1) > 0$.
Note that $0 < E(M_\infty^2)$ is always true because, via the Jensen inequality, we have

$$E(M_\infty^2) \ge (E(M_\infty))^2 = 1/4 > 0$$

Moreover, note that $E(M_\infty^2) \le E(M_\infty)$ because $0 \le M_\infty^2 \le M_\infty$.
Thus, we actually want to prove that if $E(M_\infty^2) < E(M_\infty)$ then $P(0 < M_\infty < 1) > 0$.
Suppose that $P(0 < M_\infty < 1) = 0$, then we get

$$\begin{aligned} E(M_\infty^2) &= E[M_\infty^2(\mathbf{1}(0<M_\infty<1) + \mathbf{1}(M_\infty=0) + \mathbf{1}(M_\infty=1))] = P(M_\infty=1) \\ &= E[M_\infty(\mathbf{1}(0<M_\infty<1) + \mathbf{1}(M_\infty=0) + \mathbf{1}(M_\infty=1))] = E(M_\infty) \end{aligned}$$

so we got a contradiction.
Now we will show the inequality $E(M_\infty^2) < E(M_\infty)$: by the discrete integration by parts we have

$$M_t^2 - M_{t-1}^2 = 2M_{t-1}(M_t - M_{t-1}) + (M_t - M_{t-1})^2$$

and since by the martingale property

$$E\bigg(2M_{t-1}(M_t - M_{t-1})\bigg) = E\bigg(E(2M_{t-1}(M_t - M_{t-1})|\mathcal{F}_{t-1})\bigg) = E\bigg(2M_{t-1}E(M_t - M_{t-1}|\mathcal{F}_{t-1})\bigg) = 0$$

it follows

$$E(M_t^2) = \frac{1}{4} + E\bigg(\sum_{r=1}^{t}((M_r - M_{r-1})^2\bigg) = \frac{1}{4} + \sum_{r=1}^{t}E\bigg(E\big((M_r - M_{r-1})^2\big|\mathcal{F}_{r-1}\big)\bigg)$$

Recall that $\Delta M_r = \frac{1}{r+2}(X_r - M_{r-1})$, thus

$$\begin{aligned} E\big((X_r - M_{r-1})^2\big|\mathcal{F}_{r-1}\big) &= E(X_r^2|\mathcal{F}_{r-1}) + M_{r-1}^2 - 2M_{r-1}E(X_r|\mathcal{F}_{r-1}) \\ &= E(X_r|\mathcal{F}_{r-1}) + M_{r-1}^2 - 2M_{r-1}^2 \\ &= M_{r-1} - M_{r-1}^2 \end{aligned}$$

So we have

$$E(M_t^2) = \frac{1}{4} + \sum_{r=1}^{t} \frac{1}{(r+2)^2}\Big(E(M_{r-1}) - E(M_{r-1}^2)\Big) = \frac{1}{4} + \sum_{r=1}^{t} \frac{1}{(r+2)^2}\Big(\frac{1}{2} - E(M_{r-1}^2)\Big)$$
$$< \frac{1}{4} + \frac{1}{2}\sum_{r=1}^{t} \frac{1}{(r+2)^2}$$

By Fatou's lemma

$$E(M_\infty^2) \le \liminf_{t\to\infty} E(M_t^2) \le \frac{1}{4} + \frac{1}{2}\sum_{r=1}^{\infty} \frac{1}{(r+2)^2} = \frac{1}{4} + \frac{1}{2}\sum_{r=3}^{\infty}\frac{1}{r^2} < \frac{1}{4} + \frac{1}{4} = E(M_\infty)$$

where $\sum\limits_{r=3}^{\infty} r^{-2} < \int\limits_{2}^{\infty} x^{-2}dx = 1/2$ with strict inequality.

6. Consider an i.i.d. random sequence $(U_t : t \in \mathbb{N})$ with uniform distribution on $[0,1]$, $P(U_1 \in dx) = \mathbf{1}_{[0,1]}(x)dx$. Note that $E_P(U_t) = 1/2$.

   Consider also the random variable $-\log(U_1(\omega))$ which is 1-exponential w.r.t. P.

$$P(-\log(U_1) > x) = \begin{cases} \exp(-x) & \text{kun } x \ge 0 \\ 1 & \text{kun } x < 0 \end{cases}$$

   $-\log(U_1) \in L^1(P)$ with $E_P(-\log(U_1)) = 1$.

   (a) Let $Z_0 = 1$, and

$$Z_t(\omega) = 2^t \prod_{s=1}^{t} U_s(\omega)$$

   Show that (Z_t) is a martingale in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$, with $\mathcal{F}_t = \sigma(Z_1, Z_2, \dots, Z_t) = \sigma(U_1, U_2, \dots, U_t)$.

   (b) Show that $E_P(Z_t) = 1$.

   (c) Show that the limit $Z_\infty(\omega) = \lim_{t\to\infty} Z_t(\omega)$ exists P almost surely.

   (d) Show that

$$Z_\infty(\omega) = 0 \qquad P\text{-a.s.}$$

   Hint Compute first the P-a.s. limit

$$\lim_{t\to\infty} \frac{1}{t}\log(Z_t(\omega))$$

   (remember Kolmogorov's strong law of large numbers!).

   (e) Show that the martingale $(Z_t(\omega) : t \in \mathbb{N})$ is not uniformly integrable.

   (f) Show that $\log(Z_t(\omega))$ is a supermartingale, does it satisfy the assumptions of Doob's martingale convergence theorem ?

   (g) At every time $t \in \mathbb{N}$, define the probability measure

$$Q_t(A) := E_P(Z_t \mathbf{1}_A) \qquad \forall A \in \mathcal{F}_t$$

   on the probability space $(\Omega, \mathcal{F})$.

   Show that the random variables $(U_1, \dots, U_t)$ are i.i.d. also under Q_t, compute their probability density under Q_t.

Solution

(a) Note that $0 \leq Z_t \leq 2^t \, \forall t \in \mathbb{N}$, then Z_t is integrable $\forall t \in \mathbb{N}$ and it enjoys the martingale property

$$E(Z_t|\mathcal{F}_{t-1}) = 2^t \prod_{s=1}^{t-1} U_s(\omega) E(U_t) = Z_{t-1}$$

(b) $E_P(Z_t) = E_P(Z_0) = Z_0 = 1$.

(c) $(Z_t : t \in \mathbb{N})$ is a non-negative martingale, by Doob's martingale convergence theorem $\lim_{t\to\infty} Z_t(\omega)$ exists a.s.

(d) Via the strong law of large numbers

$$\frac{1}{t}\log(Z_t) = \frac{1}{t}\left(t\log(2) + \sum_{s=1}^{t}\log(U_s)\right)$$
$$= \log(2) + \frac{1}{t}\sum_{s=1}^{t}\log(U_s) \to \log(2) + E(\log(U_1)) = \log(2) - 1 < 0.$$

with convergence P a.s., then P a.s. as $t \to \infty$ $Z_t(\omega) = \mathcal{O}(\exp(t(\log(2)-1)) \to 0$, that is $Z_\infty(\omega) = 0$.

(e) Since $E(Z_1) = 1 > E(Z_\infty) = 0$, so the martingale property does not hold at infinity, so Z_t is not uniformly integrable.

(f) By Jensen' inequality for the conditional expectation:

$$\log Z_{t-1} = \log\Big(E_P(Z_t|\mathcal{F}_{t-1})\Big) \geq E_P(\log Z_t|\mathcal{F}_{t-1})$$

therefore $\log(Z_t)$ is a supermartingale. We see that the hypothesis of the Doob convergence theorem does not hold, in fact

$$E_P(\log Z_t) = t\log 2 + \sum_{s=1}^{t} E(\log U_s) = t(\log 2 - 1) < 0$$

and

$$\sup_{t\in\mathbb{N}} E_P\big(|\log(Z_t)|\big) \geq \sup_{t\in\mathbb{N}}\Big|E_P\big(\log(Z_t)\big)\Big| = (1-\log 2)\sup_{t\in\mathbb{N}} t = +\infty$$

Therefore the collection $\{\log(Z_t) : t \in \mathbb{N}\}$ is not bounded uniformly in $L^1(P)$.

(g) For bounded measurable test functions $g_i(x)$, $i = 1, \dots, t$,

$$E_{Q_t}\big(g_1(U_1)\dots g_t(U_t)\big) = E_P\big(Z_t g_1(U_1)\dots g_t(U_t)\big) = E_P\left(2^t \prod_{s=1}^{t} U_s g_s(U_s)\right) = \prod_{s=1}^{t} E_P(2U_s g(U_s))$$
$$= \prod_{s=1}^{t} E_P(Z_s g(U_s)) = \prod_{s=1}^{t} E_P(Z_t g_s(U_s)) = \prod_{s=1}^{t} E_{Q_t}(g_s(U_s))$$

which means that $U_1, \dots, U_t$ are independent under Q_t.
For $t \in [0,1]$

$$Q(U \leq t) = E_P\big(2U\mathbf{1}(U \leq t)\big) = 2\int_0^t u du = t^2$$

and $Q(U \in dt) = 2tdt$.

7. (Paley's and Littlewood's maximal function) Consider a function in $f(x) \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dx)$. Define the σ-algebra

$$\mathcal{F}_k = \sigma\big\{ Q_{k,z} = \big(z2^{-k}, (z+1)2^{-k}\big],\ z \in \mathbb{Z}^d \big\} \subseteq \mathcal{B}(\mathbb{R}^d), \quad k \in \mathbb{Z}$$

and the two sided filtration $\mathbb{F} = (\mathcal{F}_k : k \in \mathbb{Z})$ where the dyadic cubes $(Q_{k,z} : z \in \mathbb{Z}^d)$ form a partition of $\mathbb{R}^d$, and the functions

$$f_k(x) = \sum_{z \in \mathbb{Z}^d} \mathbf{1}(x \in Q_{k,z}) \frac{1}{|Q_{k,z}|} \int_{Q_{k,z}} f(y) dy$$

where for $k \in \mathbb{Z}$, $|Q_{k,z}| = 2^{-kd}$ is the Lebesgue measure of the d-dimensional dyadic cube

(a) Show that $f_k(x)$ is an $\mathbb{F}$-martingale w.r.t. Lebesgue measure. Note that the definition of conditional expectation martingales extends directly to the case where we integrate with respect to σ-finite positive measures, where the martingale property in this case means

$$\int_{\mathbb{R}^d} f(x) g_k(x) dx = \int_{\mathbb{R}^d} f(x) g_k(x) dx$$

$\forall k \in \mathbb{Z}$ and $g_k(x)$ bounded and $\mathcal{F}_k$-measurable.
To work with a probability measure, we could take instead with $f(x) \in L^1([0,1]^d, \mathcal{B}([0,1]^d), dx)$.

(b) Show that

$$\lim_{k \to -\infty} f_k(x) = 0, \quad \forall x \in \mathbb{R}^d,$$

but it does not converge in L^1.
In particular this means that the Doob's martingale backward convergence theorem does NOT extend to the case of σ-finite measures.

(c) Show that $\lim_{k \to +\infty} f_k(x) = f(x)$ almost everywhere and in L^1.

(d) Define the *maximal function*

$$f^{\square}(x) := \sup_{k \in \mathbb{Z}} f_k(x)$$

Use the martingale maximal inequalities to show that for $1 < p < \infty$

$$\| f^{\square}(x) \|_{L^p(\mathbb{R}^d)} \leq \frac{p}{p-1} \sup_{k \in \mathbb{Z}} \| f_k \|_{L^p(\mathbb{R}^d)} \leq \frac{p}{p-1} \| f \|_{L^p(\mathbb{R}^d)}$$

and

$$cP(|f^{\square}(x)| > c) \leq \sup_{k \in \mathbb{Z}} \| f_k \|_{L^1(\mathbb{R}^d)} \leq \| f \|_{L^1(\mathbb{R}^d)}$$

Solution

(a) Note that $f_k \in \mathcal{F}_k$ and that $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$. We want to show that f_k is a martingale, so first we see that f_k is integrable $\forall k \in \mathbb{Z}$.

$$\begin{aligned}
\int_{\mathbb{R}^d} dx |f_k(x)| &= \int_{\mathbb{R}^d} dx \sum_{z \in \mathbb{Z}^d} \mathbf{1}(x \in Q_{k,z}) 2^{kd} \left| \int_{Q_{k,z}} f(y) dy \right| \\
&\leq \int_{\mathbb{R}^d} dx \sum_{z \in \mathbb{Z}^d} \mathbf{1}(x \in Q_{k,z}) 2^{kd} \int_{Q_{k,z}} |f(y)| dy \\
&\leq \sum_{z \in \mathbb{Z}^d} \int_{Q_{k,z}} |f(y)| dy = \|f\|_1 < \infty
\end{aligned}$$

where we used the monotone convergence theorem to switch the order between the sum and the integral.
Note that every $A \in \mathcal{F}_k$ is a disjoint union of squares Q_k, i.e. $A = \cup_{i\in I} Q_{k,i}$ for some set $I \subseteq \mathbb{Z}$, and each $Q_{k,i}$ is clearly the union of four squares of size $2^{-(k+1)}$. Then we can check the martingale property by using the Kolmogorov's definition of conditional expectation just for a generic Q_k

$$\begin{aligned}\int_{\mathbb{R}^d} dx f_{k+1}(x)\chi_{Q_k}(x) &= \int_{\mathbb{R}^d} dx \sum_{i:\cup_{i=1}^4 Q_{k+1,i}=Q_k} \frac{\chi_{Q_{k+1,i}}(x)}{|Q_{k+1,i}|}\int_{Q_{k+1,i}} f(y)dy = \\ &= \int_{Q_k} f(y)dy = \int_{\mathbb{R}^d} dx f_k(x)\chi_{Q_k}(x)\end{aligned}$$

(b) Note that

$$|f_k(x)| \leq 2^{kd} \parallel f \parallel_{L^1(\mathbb{R}^d)} \longrightarrow 0, \qquad \text{as } k \to -\infty$$

On the other hand we have

$$E|f_k| = \sum_{z\in\mathbb{Z}} \left| \int_{Q_{z,k}} f(x)dx \right| \to |E(f)|$$

as $k \to -\infty$, then $f_k \to f$ in L^1 as $k \to -\infty$ only if $|E(f)| = 0$

(c) To show convergence in L^1 convergence, we split the integral into two parts. Let $\epsilon > 0$. Then there exists a dyadic cube Q, such that

$$\int_{\mathbb{R}^d\setminus Q} |f(x)|dx < \epsilon$$

Then also

$$\int_{\mathbb{R}^d\setminus Q} |f_k(x) - f(x)|dx < \int_{\mathbb{R}^d\setminus Q} (|f_k(x)| + f(x)|)dx < 2\epsilon$$

because $\int_{\mathbb{R}^d\setminus Q} |f_k(x)| \leq \int_{\mathbb{R}^d\setminus Q} |f(x)|$ when k is large enough so that the cubes on the level k tile the set $\mathbb{R}^d\setminus Q$.
On the other hand, Q has finite measure (which we can normalize to be 1), and $f_k|Q$ is uniformly integrable because

$$f_k = E(f|\mathcal{F}_k)$$

Therefore $f_k|Q \to f|Q$ in L^1.
The result follows since

$$\int_{\mathbb{R}^d} |f_k(x) - f(x)|dx \leq \int_Q |f_k(x) - f(x)|dx + \int_{\mathbb{R}^d\setminus Q} |f_k(x) - f(x)|dx < 3\epsilon$$

for k large enough.
On the other hand the convergence almost surely follows from the Doob martingale convergence theorem.

(d) For $n \in \mathbb{N}$, the process $(|f_k(x)| : k = -n, \ldots, 0, \ldots, n)$ is a submartingale as one can see by the Jensen's inequality.
We want to apply the maximal Doob inequality, so we need to consider a probability measure: for each $z \in \mathbb{Z}^d$, consider the probability space $Q_{-n,z}$ equipped with the normalized Lebesgue measure $2^{-nd}dx$.
Then $\forall z \in \mathbb{Z}$, $c > 0$, by using Doob's martingale maximal inequalities

$$\begin{aligned}c\int_{Q_{-n,z}} \mathbf{1}\big(\sup_{-n\leq k\leq n} |f_k(x)| > c\big)dx &\leq \int_{Q_{-n,z}} |f_n(x)|\mathbf{1}\big(\sup_{-n\leq k\leq n} |f_k(x)| > c\big)dx \\ &\leq \int_{Q_{-n,z}} |f_n(x)|dx\end{aligned}$$

and we can sum over $z \in \mathbb{Z}^d$ obtaining

$$\begin{aligned} c\int_{\mathbb{R}^d} \mathbf{1}\big(\sup_{-n\le k\le n} |f_k(x)| > c\big)dx &\le \int_{\mathbb{R}^d} |f_n(x)|\mathbf{1}\big(\sup_{-n\le k\le n} |f_k(x)| > c\big)dx \\ &\le \int_{\mathbb{R}^d} |f_n(x)|dx \le \| f \|_{L^1(\mathbb{R})} \end{aligned} \quad (1)$$

This implies easily that

$$c\int_{\mathbb{R}^d} \mathbf{1}\big(\sup_{-n\le k\le n} |f_k(x)| > c\big)dx \le \int_{\mathbb{R}^d} |f_n(x)|dx \le \sup_{n\in\mathbb{Z}} \|f_n\|_1 \le \|f\|_1$$

Moreover, by monotone convergence we get that

$$\begin{aligned} \lim_{n\to\infty} c\int_{\mathbb{R}^d} \mathbf{1}\big(\sup_{-n\le k\le n} |f_k(x)| > c\big)dx &= c\int_{\mathbb{R}^d} \mathbf{1}\big(\sup_{k\in\mathbb{Z}} |f_k(x)| > c\big)dx \\ &\ge c\int_{\mathbb{R}^d} \mathbf{1}\big(\big|\sup_{k\in\mathbb{Z}} f_k(x)\big| > c\big)dx =: cP(|f^{\Box}(x)| > c) \end{aligned}$$

so we finally get the second inequality from 1

$$cP(|f^{\Box}(x)| > c) \le \sup_{k\in\mathbb{Z}} \| f_k \|_{L^1(\mathbb{R}^d)} \le \| f \|_{L^1(\mathbb{R}^d)}$$

Now, if $f \in L^p$, then $\|f_k\|_p \le \|f\|_p \, \forall k \in \mathbb{Z}$. Equation 1 means that the hypothesis of Lemma 20 are satisfied in the probability space specified above and for $X = \sup_{k\in\mathbb{Z}} |f_k(x)| \ge f^{\Box}(x)$ and $Y = f_n(x) \; \forall n \in \mathbb{N}$.
Thus we get

$$\int_{Q_{-n,z}} |f^{\Box}(x)|^p dx \le \left(\frac{p}{p-1}\right)^p \int_{Q_{-n,z}} |f_n(x)|^p dx$$

By summing all over $z \in \mathbb{Z}^d$ we get finally

$$\|f^{\Box}(x)\|_p \le \frac{p}{p-1}\|f_n\|_p \le \frac{p}{p-1}\sup_{k\in\mathbb{Z}} \|f_k\|_p \le \frac{p}{p-1}\|f\|_p$$