Stochastic analysis, Fall 2014, Exercises-5, 15.10.2014

1. Let $\tau(\omega) \in \mathbb{N}$ be a stopping time w.r.t. $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$. Show that

$$\mathcal{F}_\tau = \big\{A \in \mathcal{F} : A \cap \{\tau \le t\} \in \mathcal{F}_t \forall t \in \mathbb{N}\big\}$$

is a σ-algebra.

Solution

First of all, note that Ω, the universe set of $\mathcal{F}$, belongs to $\mathcal{F}_\tau$ because for all $t \in \mathbb{N}$ we have $\Omega \cap \{\tau \le t\} \in \mathcal{F}_t$.

Then we need to show the closeness of $\mathcal{F}_\tau$ with respect to the complementation: observe that if $A \in \mathcal{F}_\tau$, then $\forall t \in \mathbb{N}$

$$\begin{aligned} A \cap \{\tau \le t\} \in \mathcal{F}_t \quad & \Rightarrow\ (A \cap \{\tau \le t\})^c = A^c \cup \{\tau > t\} \in \mathcal{F}_t \\ & \Rightarrow\ \mathcal{F}_t \ni (A^c \cup \{\tau > t\}) \cap \{\tau \le t\} = (A^c \cap \{\tau \le t\}). \end{aligned}$$

For the closeness under countable unions, note that if $A_n \in \mathcal{F}_\tau$, $\forall n \in \mathbb{N}$ then $\forall t \in \mathbb{N}$

$$\left(\bigcup_{n\in\mathbb{N}} A_n\right) \cap \{\tau \le t\} = \bigcup_{n\in\mathbb{N}} (A_n \cap \{\tau \le t\}) \in \mathcal{F}_t.$$

2. We continue with the random walk. We have

$$M_t(\omega) = \sum_{s=1}^{t} X_s(\omega)$$

is a binary random walk where $t \in \mathbb{N}$ and $(X_s : s \in \mathbb{N})$ are i.i.d. random variables with

$$P(X_s = \pm 1) = P(X_s = \pm 1 | \mathcal{F}_{s-1}) = 1/2$$

X_s is $\mathcal{F}_s$ measurable and P-independent from $\mathcal{F}_{s-1}$.

Recall that $(M_t)_{t\in\mathbb{N}}$ and $(M_t^2 - t)_{t\in\mathbb{N}}$ are $\mathbb{F}$-martingales.

(a) Consider the stopping time $\tau = \tau_K = \inf\{t : M_t \ge K\}$ for $K \in \mathbb{N}$. Show that $P(\tau < \infty) = 1$.
Hint: the stopped martingale $(M_{t\wedge\tau} : t \in \mathbb{N})$ is a sub-martingale bounded from above (equivalently $(-M_{t\wedge\tau})$ is a supermartingale bounded from below).
Apply Doob forward convegence theorem,

(b) Show that P almost surely $M_\tau(\omega) = K$

(c) Show that $(M_{t\wedge\tau}(\omega) : t \in \mathbb{N})$ is not uniformly integrable.
Hint: otherwise we could interchange the expectation and the limit for $t \to \infty$ operations.

(d) Show that $E(\tau) = +\infty$
Hint: prove it by using the properties of the martingale

$$M_{t\wedge\tau}^2 - t \wedge \tau$$

Resume : a gambler plays a fair coin-toss game with unit stakes, playing from time 0 until the stopping time $\tau_K(\omega)$, when he quits the game a profit $K > 0$. With probability one $\tau_K(\omega) < \infty$, the gambler always makes a profit K which is arbitrarilty large.

This free-lunch paradox is explained as follows:

The gambler's strategy, to play until $\tau_K(\omega)$ requires an infinite amount of capital, because $\forall M \in \mathbb{N}$ $P(\tau_{-M} > \tau_K) > 0$, for any finite amount of capital there is a positive probability to lose everything before τ_K.

And even with an infinite amount of capital at disposal, although $\tau_K(\omega)$ is P a.s. finite, the expected time for winning K is $E(\tau_K) = \infty$.

Solution

(a) First, we prove that $M_{t\wedge\tau}$ is a martingale, in fact

$$M_{t\wedge\tau} = \sum_{s=1}^{t} \chi(s \leq \tau)(M_s - M_{s-1})$$

Note that $\{s \leq \tau\} = \{\tau > s\}^c = \{\tau \geq s-1\}^c \in \mathcal{F}_{s-1}$, hence $\chi(s \leq \tau)$ is a predictable process.
Moreover $E|\chi(s \leq \tau)(M_s - M_{s-1})| \leq 1$, hence, via the theorem about the martingale transform, we get that $M_{t\wedge\tau}$ is a martingale. Note that this proof holds for every stopped martingale since we did not use the special form of our martingale M_t.
Moreover, observe that $M_{t\wedge\tau}$, a fortiori, is a submartingale bounded from above since $M_{t\wedge\tau} \leq K$. Then

$$\begin{aligned} E(|M_{t\wedge\tau}|) &= -E(M_{t\wedge\tau}) + 2E(M^+_{t\wedge\tau}) \\ &\leq -E(M_0) + 2E(M^+_{t\wedge\tau}) \leq 2K \end{aligned} \tag{1}$$

Therefore we can apply the Doob convergence theorem which says that almost surely there exists the limit

$$\lim_{t\to\infty} M_{t\wedge\tau} = M_\tau$$

and $M_\tau \in L^1$. Now, by looking at the proof of the Doob convergenze theorem we have that

$$P\left(\bigcup_{a\in\mathbb{Z}} \{U_{[a,a+1]}([0,\infty)) = \infty\}\right) = 0$$

i.e. the oscillation of the process are finite almost surely, which means that, since the process is discrete and we cannot have any asymptotic behaviour, the process reaches its limit in finite time almost surely, therefore $P(\tau < \infty) = 1$.

(b) Note that $\tau \geq K$, so we can write

$$\begin{aligned} M_\tau(\omega) &= M_\tau(\omega) \sum_{m=0}^{\infty} \chi_{\{\tau=m\}}(\omega) \\ &= \sum_{m=0}^{\infty} M_m(\omega)\chi_{\{\tau=m\}}(\omega) \\ &= \sum_{m=0}^{K-1} M_m(\omega)\chi_{\{\tau=m\}}(\omega) + \sum_{m=K}^{\infty} M_m(\omega)\chi_{\{\tau=m\}}(\omega) \\ &= \sum_{m=0}^{K-1} M_m(\omega)\chi_{\{\tau=m\}}(\omega) + K\sum_{m=K}^{\infty} \chi_{\{\tau=m\}}(\omega) \end{aligned}$$

But $P(\tau < K) = 0$, then almost surely it holds $M_\tau(\omega) = K$.

(c) By using the hint we observe if $M_{t\wedge\tau}$ were uniformly integrable then the optional stopping time theorem says that

$$E(M_{t\wedge\tau}) = E(M_0).$$

But $E(M_\tau) = K$ and $E(M_0) = 0$, so we have a contradiction.

(d) By the martingale property of $M^2_{t\wedge\tau} - t\wedge\tau$, we get $\forall t \in \mathbb{N}$

$$E(M^2_{t\wedge\tau}) = E(t\wedge\tau) \le E(\tau)$$

and then

$$\sup_t E(M^2_{t\wedge\tau}) = \sup_t E(t\wedge\tau) = E(\tau)$$

If $E(\tau)$ were finite, then also $\sup_t E(M^2_{t\wedge\tau})$ would be finite, but we know that if for some $p > 1$

$$\sup_t E(|M_t|^p) < \infty$$

then the sequence $\{M_t : t \in \mathbb{N}\}$ is uniformly integrable, which is a contradiction.

3. A three-player ruin problem: Initially, three players have respectively $a, b, c \in \mathbb{N}$ units of capital. Games are independent and each game consists of choosing two players at random (i.e. uniformly) and transferring one unit from the first-chosen to the second-chosen player. Once a player is ruined, he is ineligible for further play.

Let τ_1 be the number of games required for one player to be ruined, and let τ_2 be the number of games required for two players to be ruined.

Let (X_t, Y_t, Z_t) be the numbers of units possessed by the three players after the t-game, and

$$M_t := X_tY_tZ_t + \frac{(a+b+c)t}{3} \quad \text{and}$$
$$N_t := X_tY_t + X_tZ_t + Y_tZ_t + t$$

(a) Show that the stopped processes $(M_{t\wedge\tau_1} : t \in \mathbb{N})$ and $(N_{t\wedge\tau_2} : t \in \mathbb{N})$ are non-negative $\mathbb{F}$-martingales where $\mathcal{F}_t = \sigma(X_s, Y_s, Z_s, s \le t)$.

(b) Use Doob martingale convergence theorem and Fatou lemma to show that $E(\tau_k) < \infty$, for $k = 1, 2$

(c) Knowing that $E(\tau_k) < \infty$,
show that $(M_{t\wedge\tau_1} : t \in \mathbb{N})$ and $(N_{t\wedge\tau_2} : t \in \mathbb{N})$ are uniformly integrable.

(d) Use uniform integrability of the stopped martingales $(M_{t\wedge\tau_1} : t \in \mathbb{N})$ and $(N_{t\wedge\tau_2} : t \in \mathbb{N})$ to compute $E(\tau_k)$ for $k = 1, 2$.

Solution

(a) We check the martingale property for the process M_t on the set $\{\omega : \tau_1(\omega) > t\}$:

$$\begin{aligned} M_t &= (X_{t-1} + \Delta X_t)(Y_{t-1} + \Delta Y_t)(Z_{t-1} + \Delta Z_t) + \frac{(a+b+c)t}{3} \\ &= M_{t-1} + \Delta X_t Y_{t-1} Z_{t-1} + \Delta Y_t X_{t-1} Z_{t-1} + \Delta Z_t Y_{t-1} X_{t-1} \\ &\quad + \Delta X_t \Delta Y_t Z_{t-1} + \Delta Y_t \Delta Z_t X_{t-1} + \Delta Z_t \Delta X_t Y_{t-1} \\ &\quad + \Delta X_t \Delta Y_t \Delta Z_t + \frac{t(a+b+c)}{3} \end{aligned}$$

Now, since the games are independent, we have

$$E(\Delta X_t \Delta Y_t \Delta Z_t | \mathcal{F}_{t-1}) = E(\Delta X_t \Delta Y_t \Delta Z_t) = 0$$
$$E(\Delta X_t \Delta Y_t Z_{t-1} | \mathcal{F}_{t-1}) = Z_{t-1} E(\Delta X_t \Delta Y_t) = -\frac{Z_{t-1}}{3}$$
$$E(\Delta X_t Y_{t-1} Z_{t-1} | \mathcal{F}_{t-1}) = Y_{t-1} Z_{t-1} E(\Delta X_t)$$

By using the symmetry and noting that $X_t + Y_t + Z_t = a + b + c$ we get

$$E(\Delta M_t | \mathcal{F}_{t-1}) = 0,$$

then $M_{t\wedge\tau_1}$ is a martingale. In addition, $M_{t\wedge\tau_1}$ is non-negative since

$$X_{t\wedge\tau_1} Y_{t\wedge\tau_1} Z_{t\wedge\tau_1} \geq 0$$

Similarly for N_t on the set $\{\omega : \tau_1(\omega) > t\}$ we have

$$\begin{aligned} N_t &= (X_{t-1} + \Delta X_t)(Y_{t-1} + \Delta Y_t) + (X_{t-1} + \Delta X_t)(Z_{t-1} + \Delta Z_t) \\ &\quad + (Y_{t-1} + \Delta Y_t)(Z_{t-1} + \Delta Z_t) + t \\ &= N_{t-1} + \Delta X_t \Delta Y_t + \Delta Y_t \Delta Z_t + \Delta X_t \Delta Z_t \\ &\quad + Y_{t-1}\Delta X_t + Y_{t-1}\Delta Z_t + X_{t-1}\Delta Y_t + X_{t-1}\Delta Z_t + Z_{t-1}\Delta X_t + Z_{t-1}\Delta Y_t + 1 \end{aligned}$$

Note that

$$\begin{aligned} &E(X_{t-1}\Delta Y_t | \mathcal{F}_{t-1}) = X_{t-1} E(\Delta Y_t) = 0 \\ &E(\Delta X_t \Delta Y_t) = -\frac{1}{3} \end{aligned}$$

Then, by using the symmetries, we get

$$E(\Delta N_t | \mathcal{F}_{t-1}) = 0$$

hence, also $N_{t\wedge\tau_2}$ is a martingale. Furthermore, $N_{t\wedge\tau_2}$ is also non negative since

$$X_{t\wedge\tau_2} Y_{t\wedge\tau_2} + X_{t\wedge\tau_2} Z_{t\wedge\tau_2} + Y_{t\wedge\tau_2} Z_{t\wedge\tau_2} \geq 0.$$

(b) By Doob convergence theorem we get that almost surely

$$M_{t\wedge\tau_1} \longrightarrow M_{\tau_1} \in L^1$$

and

$$N_{t\wedge\tau_2} \longrightarrow N_{\tau_2} \in L^1$$

But by definition $M_{\tau_1} = \frac{a+b+c}{3}\tau_1$ and $N_{\tau_2} = \tau_2$, hence $\tau_1, \tau_2 \in L^1$

(c) Note that

$$M_{t\wedge\tau_1} \leq (a+b+c)^3 + \frac{a+b+c}{3}\tau_1 \in L^1$$

and

$$N_{t\wedge\tau_2} \leq 3(a+b+c)^2 + \tau_2 \in L^1$$

thus both the sequences $M_{t\wedge\tau_1}, N_{t\wedge\tau_2} : t \in \mathbb{N}$ are uniformly integrable since they are uniformly bounded by an integrable random variable.

(d) Since $M_{t\wedge\tau_1}$ and $N_{t\wedge\tau_2}$ are uniformly integrable martingales, the Doob optional stopping theorem applies:

$$abc = E(M_0) = E(M_{\tau_1}) = \frac{a+b+c}{3} E(\tau_1)$$

hence

$$E(\tau_1) = \frac{3abc}{a+b+c}.$$

Similarly for τ_2 we have

$$ab + bc + ac = E(N_0) = E(N_{\tau_2}) = E(\tau_2)$$

4. A generalization of a game by Jacob Bernoulli. In this game a fair dice is rolled, and if the result is Z_1, then Z_1 dice are rolled. If the total of the Z_1 dice is Z_2, then Z_2 dice are rolled. If the total of the Z_2 dice is Z_3, then Z_3 dice are rolled, and so on. Let $Z_0 \equiv 1$.
Find a positive constant α such that

$$M_t(\omega) = Z_t(\omega)\alpha^t \quad t \in \mathbb{N}$$

is a $\mathbb{F}$-martingale where $\mathcal{F}_t = \sigma(Z_0, Z_1, \ldots, Z_t)$. Hint: compute $E(Z_{t+1}|\mathcal{F}_t)$.
What does Doob's martingale convergence theorem tell us about this?

Solution
By following the hint we consider

$$E(Z_t|\mathcal{F}_{t-1}) = E(\sum_{i=1}^{Z_{t-1}} U_{t,i}|\mathcal{F}_{t-1})$$

where $(U_{t,i} : t, i \in \mathbb{N})$ are i.i.d. uniformly distributed on $\{1, 2, 3, 4, 5, 6\}$. Now

$$\begin{aligned}
E\left(\sum_{i=1}^{Z_{t-1}(\omega)} U_{t,i}(\omega) \sum_{m=0}^{6^{t-1}} \chi(Z_{t-1} = m)|\mathcal{F}_{t-1}\right) &= \sum_{m=0}^{6^{t-1}} E\left(\sum_{i=1}^{Z_{t-1}(\omega)} U_{t,i}(\omega)\chi_{(Z_{t-1}=m)}(\omega)|\mathcal{F}_{t-1}\right) \\
&= \sum_{m=0}^{6^{t-1}} E\left(\sum_{i=1}^{m} U_{t,i}\right) \chi_{(Z_{t-1}=m)}(\omega) \\
&= \sum_{m=0}^{6^{t-1}} mE\left(U_{1,1}\right) \chi_{(Z_{t-1}=m)}(\omega) \\
&= E\left(U_{1,1}\right) \sum_{m=0}^{6^{t-1}} m\chi_{(Z_{t-1}=m)}(\omega) \\
&= E\left(U_{1,1}\right) Z_{t-1}(\omega)
\end{aligned}$$

This means that the constant α we are looking for is $\alpha = (E(U_{1,1}))^{-1}$.
Note that, by Doob martingale convergence theorem, we get that the limit $\lim_{t\to\infty} M_t =: M_\infty$ exists and it is integrable.

5. (a) If $(M_t(\omega) : t \in \mathbb{N})$ is a $\mathbb{F}$-martingale and $f(x)$ is convex such that $E(|f(X_t)|) < \infty \forall t \in \mathbb{N}$, show that $(f(M_t(\omega)) : t \in \mathbb{N})$ is an $\mathbb{F}$-submartingale.
   (b) If $(M_t(\omega) : t \in \mathbb{N})$ is a $\mathbb{F}$-submartingale and $f(x)$ is convex non-decreasing such that $E(|f(X_t)|) < \infty \forall t \in \mathbb{N}$, show that $(f(M_t(\omega)) : t \in \mathbb{N})$ is an $\mathbb{F}$-submartingale. Hint: use Jensen inequality for conditional expectation.

Solution

(a) By using the Jensen inequality for conditional expectations we get

$$E(f(M_t)|\mathcal{F}_{t-1}) \geq f(E(M_t|\mathcal{F}_{t-1}) = f(M_{t-1}).$$

(b) As in the former case we have

$$E(f(M_t)|\mathcal{F}_{t-1}) \geq f(E(M_t|\mathcal{F}_{t-1}) \geq f(M_{t-1})$$

where in the last inequality we used the monotonicity of f.