Stochastic analysis, spring 2014, Exercises-4, 8.10.2014

1. Let $\tau_1(\omega)$ and $\tau_2(\omega)$ stopping times with respect to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in T)$ taking values in T. Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.
   Use the definition of stopping time to show that $\sigma(\omega) = \min(\tau_1(\omega), \tau_2(\omega))$ is a $\mathbb{F}$-stopping time.

   Solution
   We have
   $$\{\sigma(\omega) \le t\} = \{\omega : \tau_1(\omega) \le t\} \cup \{\omega : \tau_2(\omega) \le t\} \in \mathcal{F}_t$$
   so $\sigma(\omega)$ is a stopping time.

2. Let $(M_t(\omega))_{t\in\mathbb{N}}$ a martingale with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $M_0(\omega) = 0$. Define the family of random times $\tau_x : x \in \mathbb{R}$
   $$\tau_x(\omega) = \begin{cases} \inf\{s : M_s \ge x\} & \text{for } x \ge 0 \\ \inf\{s : M_s \le x\} & \text{for } x < 0 \end{cases}$$
   Show that τ_x is a stopping time.

   Solution
   Assume $x \ge 0$, then
   $$\{\tau_x \le t\} = \{\omega : \inf\{s : M_s(\omega) \ge x\} \le t\} = \cup_{s=0}^t \{\omega : M_s(\omega) \ge x\}$$
   where $\{\omega : M_s(\omega) \ge x\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$, so $\{\tau_x \le t\} \in \mathcal{F}_t$ and τ_x is a stopping time.
   The case $x \le 0$ is analogous.

3. Consider a symmetric random walk in discrete time,
   $$M_n = X_1 + \dots + X_n$$
   where $(X_k : k \in \mathbb{N})$ are independent and identically distributed Bernoulli random variables with $P(X_n = 1) = P(X_n = -1) = 1/2$.

   (a) Compute $P(M_n = k)$ for $n, k \in \mathbb{N}$.

   (b) For $x \in \mathbb{R}$, use Stirling approximation of the factorial of a large $n \in \mathbb{N}$
   $$n! \sim \exp(-n) n^n \sqrt{2\pi n}$$
   to approximate
   $$P\big(M_{2n} = 2\lfloor x \rfloor\big)$$

   (c) Consider the filtration generated by the random walk $\mathbb{F} = \big(\mathcal{F}_n^X\big)$, with $\mathcal{F}_n^X = \sigma(X_k : 0 \le k \le n)$. Show that
   $$M_n, \quad (M_n^2 - n), \text{ and } \exp\big(-\theta M_n\big)\cosh(\theta)^{-n}$$
   are $(P, \mathbb{F})$-martingales, where $\cosh(x) = \big(e^x + e^{-x}\big)/2$.

   (d) Prove the *Markov property*
   $$P\big(M_n = k \big| \mathcal{F}_{n-1}\big)(\omega) = P\big(M_n \big| M_{n-1}\big)(\omega) = P\big(X_n = k - \ell\big)\Big|_{\ell = M_{n-1}(\omega)}$$
   and, for $0 \le m \le n$
   $$P\big(M_n = k \big| \mathcal{F}_m\big)(\omega) = P\big(M_n \big| M_m\big)(\omega) = P\big(M_{n-m} = k - \ell\big)\Big|_{\ell = M_m(\omega)}$$

(e) In discrete time, let τ be a stopping time with respect to the filtration $\mathbb{F}$, then the stopped σ-algebra $\mathcal{F}_\tau$ is defined as

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : \forall t \in \mathbb{N},\ A \cap \{\tau \le t\} \in \mathcal{F}_t\}$$

Show that τ itself is $\mathcal{F}_\tau$-measurable. Hint: use the definition of stopping time.

(f) Show the *strong Markov property* of the random walk:

$$\widetilde{M_n} := (M_{\tau+n} - M_\tau)$$

is a symmetric random walk independent from the stopped σ-algebra $\mathcal{F}_\tau$. Hint:

$$A = \bigcup_{k \in \mathbb{N}} A \cap \{\tau = k\}$$

and A is $\mathcal{F}_\tau$-measurable if and only if $\forall k$, $A \cap \{\tau = k\}$ is $\mathcal{F}_k$ measurable. Use the definition of conditional expectation w.r.t. $\mathcal{F}_\tau$.

(g) Consider the stopping time $\sigma(\omega) = \min(\tau_a, \tau_b)$ where $a < 0 < b \in \mathbb{N}$, and the stopped martingales $(M_{t\wedge\sigma})_{t\in\mathbb{N}}$ and $(M^2_{t\wedge\sigma} - t \wedge \sigma)_{t\in\mathbb{N}}$.
Show that Doob's martingale convergence theorem applies and

$$\lim_{t\to\infty} M_{t\wedge\sigma}(\omega) = M_\sigma(\omega)$$

exists P-almost surely.

(h) Consider now $(M^2_{t\wedge\sigma} - t \wedge \sigma)$. Use the martingale property together with the reverse Fatou lemma to show that $E(\sigma) < \infty$ which implies $P(\sigma < \infty) = 1$.

(i) For $a < 0 < b \in \mathbb{N}$, compute $P(\tau_a < \tau_b)$.
Hint: a martingale has constant expectation $E_P(M_t) = E_P(M_0)$. This holds also for the stopped martingale $M^\tau_t = M_{t\wedge\tau}$.

Solution

(a) Let denote by m the number of X's taking value 1. When $M_n = k$ we have necessarily that

$$k = m - (n - m)$$

therefore we have $m = (k + n)/2$ random variables X taking value 1 and $(n - k)/2$ taking value -1. This means that

$$P(M_n = k) = \left(\frac{1}{2}\right)^{\left(\frac{n+k}{2}\right)} \left(\frac{1}{2}\right)^{\left(\frac{n-k}{2}\right)} \binom{n}{\left(\frac{n+k}{2}\right)} = 2^{-n} \frac{n!}{\left(\frac{k+n}{2}\right)!\left(\frac{n-k}{2}\right)!}$$

Note that if n is even, the previous formula holds only if k is even as well, otherwise $P(M_n = k) = 0$ and, conversely, is n is odd the previous formula holds only if k is odd, otherwise $P(M_n = k) = 0$.

(b) By using the Stirling's formula we get

$$\begin{aligned} P(M_{2n} = 2\lfloor x \rfloor) &\simeq 2^{-2n} \frac{e^{-2n}(2n)^{2n}\sqrt{4\pi n}}{(n + \lfloor x \rfloor)^{n+\lfloor x \rfloor}(n - \lfloor x \rfloor)^{n-\lfloor x \rfloor}\sqrt{4\pi(n^2 - \lfloor x \rfloor^2)}} = \\ &= n^{2n}\sqrt{\frac{n}{n^2 - \lfloor x \rfloor^2}} \frac{(n-k)^k}{(n+k)^k(n^2 - k^2)^n} \end{aligned}$$

(c) Clearly M_n and $M_n^2 - n$ are bounded by n and n^2-1, so they are integrable. They are also martingales since if $m < n$, then

$$E(M_n|\mathcal{F}_m) = E(M_m + X_{m+1} + \cdots + X_n|\mathcal{F}_m) = M_m$$

by independence. Similarly for $M_n^2 - n$ we have

$$\begin{aligned} E(M_n^2 - n|\mathcal{F}_m) &= E((M_m + \sum_{i=m+1}^{n} X_i)^2 - n|\mathcal{F}_m) = \\ &= E(M_m^2 + 2M_m(\sum_{i=m+1}^{n} X_i) + (\sum_{i=m+1}^{n} X_i)^2 - n|\mathcal{F}_m) = \\ &= M_m^2 - n + \sum_{i,j=m+1}^{n} E(X_i X_j|\mathcal{F}_m) = \\ &= M_m^2 - n + \sum_{i,j=m+1}^{n} E(X_i X_j) = M_m^2 - m \end{aligned}$$

And finally for $e^{-\theta M_n (\cosh\theta)^{-n}}$ we get

$$\begin{aligned} E(e^{-\theta M_n}(\cosh\theta)^{-n}|\mathcal{F}_m) &= (\cosh\theta)^{-n} E(e^{-\theta(M_m + \sum_{i=m+1}^{n} X_i)}|\mathcal{F}_m) \\ &= (\cosh\theta)^{-n} e^{-\theta M_m} \prod_{i=m+1}^{n} E(e^{-\theta X_i}) = (\cosh\theta)^{-m} e^{-\theta M_m} \end{aligned}$$

(d) By using the independence we get

$$\begin{aligned} P(M_n = k|\mathcal{F}_{n-1})(\omega) &= P(M_{n-1} + X_n = k|\mathcal{F}_{n-1})(\omega) \\ &= P(\ell + X_n = k)|_{\ell = M_{n-1}} \\ &= P(X_n = k - \ell)_{\ell = M_{n-1}} \end{aligned}$$

and similarly we get

$$\begin{aligned} P(M_n = k|\mathcal{F}_m)(\omega) &= P(M_m + \sum_{i=m+1}^{n} X_i = k|\mathcal{F}_m)(\omega) \\ &= P(M_m + \hat{M}_{n-m} = k|\mathcal{F}_m) \\ &= P(\hat{M}_{n-m} = k - \ell)_{\ell = M_m} = P(M_{n-m} = k - \ell)_{\ell = M_m} \end{aligned}$$

where $\hat{M}_{n-m}$ is a copy of the random variable M_{n_m} independent from M_m.

(e) Note that $\tau \in \mathcal{F}_\tau$ if and only if $\{\omega : \tau(\omega) \le s\} \in \mathcal{F}_\tau, \forall\, s \in \mathbb{N}$.
Consider the set $\mathcal{F} \ni A := \{\omega : \tau(\omega) \le s\}$. According to the definition of $\mathcal{F}_\tau$, we want to show that for all $s, t \in \mathbb{N}$ we have

$$A \cap \{\tau \le t\} \in \mathcal{F}_t.$$

In fact, for all $s, t \in \mathbb{N}$ we have

$$A \cap \{\tau \le t\} = \{\tau \le s\} \cap \{\tau \le t\} \in \mathcal{F}_{s \wedge t} \subset \mathcal{F}_t$$

(f) Consider a bounded test function f and $A \in \mathcal{F}_\tau$, then

$$\begin{aligned} E(f(\tilde{M}_n)\chi(A)) &= E\left(f(\tilde{M}_n)\chi(A)\sum_{k=0}^{\infty}\chi(\tau = k)\right) \\ &= \sum_{k=0}^{\infty} E\left(f(\tilde{M}_n)\chi(A)\chi(\tau = k)\right) \end{aligned}$$

where we used the dominated convergence theorem and we observe that $\chi(A)\chi(\tau = k) = \chi(A \cap \{\tau = k\}) \in \mathcal{F}_k$. Thus, we have

$$\begin{aligned}
E(f(\tilde{M}_n)\chi(A)) &= \sum_{k=0}^{\infty} E\left(f(M_{n+k} - M_k)\chi(A \cap \{\tau = k\})\right) \\
&= \sum_{k=0}^{\infty} E\left(f(\hat{M}_n)\chi(A \cap \{\tau = k\})\right) \\
&= E\left(f(\hat{M}_n)\sum_{k=0}^{\infty} \chi(A \cap \{\tau = k\})\right) \\
&= E(f(\hat{M}_n))P(A) = E(f(M_n))P(A)
\end{aligned}$$

where $\hat{M}_n$ is a copy of M_n independent from τ.

(g) We recall that the Doob's martingale convergence theorem says that, given a supermartingale $X_t : t \in \mathbb{N}$ with

$$\sup_{t\in\mathbb{N}} E_P(X_t^-) < \infty,$$

then P-almost surely there exist the limit

$$\lim_{t\to\infty} X_t(\omega) = X_\infty(\omega)$$

with $X_\infty(\omega) \in L^1(\Omega)$.
Hence, in order to show the existence of the limit, we just need to check that

$$\sup_{t\in\mathbb{N}} E_P(M_{t\wedge\sigma}^-) < \infty$$

which is clear because $M_{t\wedge\sigma}^- \leq -a$.
For $M_{t\wedge\sigma}^2 - t\wedge\sigma$, the martingale property implies

$$E(M_{t\wedge\sigma}^2 - t\wedge\sigma)^- = E(M_{t\wedge\sigma}^2 - t\wedge\sigma)^+ - E(M_0^2)$$

then if we get a bound for $\sup_{t\geq 0} E(M_{t\wedge\sigma}^2 - t\wedge\sigma)^+$ we get a bound for $(M_{t\wedge\sigma}^2 - t\wedge\sigma)^-$ since $E(M_0^2) = 0$.
But we have that

$$(M_{t\wedge\sigma}^2 - t\wedge\sigma)^+ \leq a^2 + b^2$$

hence $M_{t\wedge\sigma}^2 - t\wedge\sigma$ converges almost surely to $M_\sigma^2 - \sigma$, so the claim is proved.

(h) We know that $M_{t\wedge\sigma}^2 - t\wedge\sigma$ is a martingale that converges almost surely to $M_\sigma^2 - \sigma$, so by reverse Fatou lemma

$$E(M_\sigma^2 - \sigma) \geq \limsup_{t\to\infty} E(M_{t\wedge\sigma}^2 - t\wedge\sigma) = 0,$$

therefore it follows that

$$E(\sigma) \leq E(M_\sigma^2) < \infty$$

(i) By the bounded convergence theorem we get

$$\lim_{t\to\infty} E(M_{\sigma\wedge t}) = E(M_\sigma) = E[M_\sigma(\chi(\tau_a < \tau_b) + \chi(\tau_a > \tau_b))] = P(\tau_a < \tau_b)a + (1 - P(\tau_a < \tau_b))b$$

but $E(M_\sigma) = E(M_0) = 0$, then

$$P(\tau_a < \tau_b) = \frac{b}{b-a}$$

4. Let $M_t(\omega) = B_t(\omega)$, $t \in \mathbb{R}^+$, a Brownian motion which is assumed to be $\mathbb{F}$-adapted, and such that for all $0 < s < t$ the increment $(B_t - B_s)$ is P-independent from the σ-algebra $\mathcal{F}_s$.

Note this since by assumption the Brownian motion is $\mathbb{F}$-adapted, it follows that $\mathcal{F}_t^B = \sigma(B_s : 0 \leq s \leq t) \subseteq \mathcal{F}_t$, which could be strictly bigger.

We have seen in the lectures that

$$B_t,\ M_t = (B_t^2 - t) \text{ and } Z_t = \exp(aB_t - a^2 t/2)$$

are $(P, \mathbb{F})$-martingales.

(a) Let $\sigma(\omega) = \min(\tau_a(\omega), \tau_b(\omega))$, for $a < 0 < b \in \mathbb{R}$. We will see in the lectures that the Doob martingale convergence theorem applies also to continuous martingales in continuous time. By following the same line of proof as in the random walk case check that $P\big(\sigma < \infty\big) = 1$.

(b) Let $a < 0 < b \in \mathbb{R}$. Compute $P\big(\tau_a < \tau_b\big)$.
Hints: When M is either a Brownian motion or a random walk, the stopped process $M_{t\wedge\sigma}(\omega)$ is a uniformly bounded martingale. To compute $P(\tau_a < \tau_b)$, use first the martingale property

$$E(M_{t\wedge\sigma}) = E(M_0) = 0,$$

then for $t \to \infty$ use the bounded convergence theorem.

(c) Use Doob martingale convergence theorem to show that $Z_\infty = \lim_{t\to\infty} Z_t(\omega)$ exists P almost surely.

(d) Show that $Z_\infty(\omega) = 0$ P-a.s.
Hint: Use the strong law of large numbers to show that

$$\lim_{t\to\infty} \log(Z_t)/t = -1/2,\ P \text{ a.s.}$$

Solution

(a) Exactly as in the discrete time case, by using the martingale convergence theorem and the reverse Fatou lemma for the martingale $B_{t\wedge\sigma}^2 - (t \wedge \sigma)$ we get

$$E(\sigma) \leq E(B_\sigma^2) < \infty$$

and hence $P(\sigma < \infty) = 1$

(b) Again, like in the previous exercise we get

$$E(B_\sigma) = E(B_0) = P(\tau_a < \tau_b)a + (1 - P(\tau_a < \tau_b))b$$

so that

$$P(\tau_a < \tau_b) = \frac{b}{b-a}$$

(c) Since $\sup_t E(Z_t^-) = 0$, the martingale convergence theorem tells us that there exists the limit Z_∞ with probability 1.

(d) Consider

$$C(t) := \log(Z_t)/t = aB_t/t - a^2/2$$

By the strong law of large numbers we know that

$$\lim_{t\to\infty} B_t/t = 0 \ P \text{ a.s.}$$

then we get

$$\lim_{t\to\infty} C(t) = -a^2/2 \ P \text{ a.s.}$$

Finally

$$Z_\infty(\omega) = \lim_{t\to\infty} e^{-a^2 t/2} = 0 \ P \text{ a.s.}$$

Note that for any t we have $E(Z_t)$=1, in fact In fact we have

$$\begin{aligned}
E(z_t) = E(e^{aB_t - a^2 t/2}) &= e^{-a^2 t/2} E(e^{aB_t}) = \\
&= e^{-a^2 t/2} E\left(\sum_{k=1}^{\infty} \frac{1}{k!} a^k B_t^k\right) = \\
&= e^{-a^2 t/2} \sum_{k=1}^{\infty} \frac{1}{k!} a^k \frac{(2k)!}{2^k k!} t^k = \\
&= e^{-a^2 t/2} \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{ta^2}{2}\right)^k = 1
\end{aligned}$$

This shows that the L^1-limit is not Z_∞ which is, according to the martingale convergence theorem, just the limit with probability 1.