Stochastic analysis, fall 2014, Exercises-3, 01.10.14

1. Let $(\dot\eta_n(t) : n \in \mathbb{N})$ an othonormal system in $L^2([0,1], dt)$, with

$$\int_0^1 \dot\eta_n(s)\dot\eta_m(s)ds = \delta_{n,m}$$

for example the Haar system we have used in Paul Lévy's construction of Brownian motion, and let $(G_n(\omega) : n \in \mathbb{N})$ a sequence of i.i.d. Gaussian random variable with $E(G) = 0$ and $E(G^2) = 1$. show that the random functions

$$\sum_{k=1}^n \dot\eta_k(s)G_k(\omega)$$

do not converge in $L^2(\Omega \times [0,1], dP \otimes dt)$. Hint: compute the squared norm

$$E\biggl(\int_0^1 \biggl\{\sum_{k=1}^N G_k(\omega)\dot\eta_k(s)\biggr\}^2 ds\biggr)$$

Solution
Note that by triangle inequality one can show that if the sequence $f_n \in L^2(\mu)$ for some measure μ and f is its $L^2(\mu)$ limit, then

$$\lim_{n\to\infty} \|f_n\|_{L^2(\mu)} = \|f\|_{L^2(\mu)}.$$

Consider

$$\begin{aligned}
E\biggl(\int_0^1 \biggl\{\sum_{k=1}^N G_k(\omega)\dot\eta_k(s)\biggr\}^2 ds\biggr) &= E\biggl(\int_0^1 \biggl\{\sum_{k,m=1}^N G_k(\omega)G_m(\omega)\dot\eta_k(s)\dot\eta_m(s)\biggr\}ds\biggr) = \\
&= E\biggl(\sum_{k,m=1}^N G_k(\omega)G_m(\omega)\delta_{mk}\biggr) = \\
&= E\biggl(\sum_{k=1}^N G_k^2(\omega)\biggr) = N
\end{aligned}$$

which diverges as $N \to \infty$.
This means that $\sum_{k=1}^n \dot\eta_k(s)G_k(\omega)$ does not converge in $L^2(\Omega \times [0,1], dP \otimes dt)$.

2. Let $\xi(\omega) = (\xi_1(\omega), \dots, \xi_d(\omega)) \in \mathbb{R}^d$ a Gaussian random vector with independent and identically distributed components standard Gaussian components $\xi_k(\omega) \sim \mathcal{N}(0,1)$, $E_P(\xi_k) = 0$ ja $E_P(\xi_k\xi_\ell) = \delta_{k\ell}$. Let $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{R}^d$ a deterministic vector and $A = (A_{ij} : 0 \le i \le j \le d)$ a deterministic $d \times d$ matrix.

   Let $X(\omega) = (\mu + A\xi(\omega)^\top) \in \mathbb{R}^d$.

   (a) Show that $E_P(X) = \mu$ ja $E_P(X_iX_j) - E_P(X_i)E_P(X_j) = \Sigma_{ij}$, where $\Sigma = AA^\top$.

   (b) Show that the random vector X has density with respect to the Lebesgue measure in $\mathbb{R}^d$ given by

$$p_X(x) = (2\pi)^{-d/2}\det(\Sigma)^{-1/2}\exp\Bigl(-\frac{1}{2}(x-\mu)\Sigma^{-1}(x-\mu)^\top\Bigr)$$

in other words, if $g : \mathbb{R}^d \to \mathbb{R}$ is

$$E_P\Big(g(X)\Big) = E_P\big(g(\mu + A\xi^\top)\big) =$$

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g\big(\mu_1 + \sum_{j=1}^d A_{1j} y_j, \ldots, \mu_d + \sum_{j=1}^d A_{1j} y_j\big) \prod_{j=1}^d \Big\{ \frac{1}{\sqrt{2\pi}} \exp(-y_j^2/2) \Big\} dy_1 \ldots dy_d =$$

$$\int_{\mathbb{R}^d} g(x_1, \ldots, x_d) p_X(x_1, \ldots, x_d) dx_1 \ldots dx_d$$

Hint use the change of variables $x = \mu + Ay^\top$. You can assume that $m = n$ and the matrix A is invertible.

Solution:

(a) Componentwise we have

$$E_P(X_i) = E_P\Big(\mu_i + \sum_j A_{ij}\xi_j\Big) = \mu_i + \sum_j A_{ij} E_P \xi_j = \mu_i$$

$$\begin{aligned}
E_P(X_i X_j) - E_P(X_i)E_P(X_j) &= E_P(\mu_i + \sum_j A_{ik}\xi_k)(\mu_j + \sum_m A_{jm}\xi_m) - \mu_i\mu_j = \\
&= \sum_{k,m} A_{ik} A_{jm} E_P(\xi_k \xi_m) = \\
&= \sum_{k,m} A_{ik} A_{jm} \delta_{km} = \\
&= \sum_k A_{ik} A_{jk} = \\
&= \sum_k A_{ik} A^\top_{kj} = \Sigma_{ij}
\end{aligned}$$

(b) Consider

$$E_P\Big(g(X)\Big) = E_P\big(g(\mu + A\xi^\top)\big) = \int_{\mathbb{R}^d} g\big(\mu_1 + \sum_{j=1}^d A_{1j} y_j, \ldots\big) \prod_{j=1}^d \Big\{ \frac{1}{\sqrt{2\pi}} \exp(-y_j^2/2) dy_i \Big\}$$

We use the change of variables $x = \mu + Ay^\top$, so that the Jacobian matrix is $\partial y_i / \partial x_j = A^{-1}_{ij}$. Then we have

$$\begin{aligned}
E_P\Big(g(X)\Big) &= \int_{\mathbb{R}^d} |\det A^{-1}| g\big(x_1, \ldots, x_d\big)(2\pi)^{-d/2} \times \\
&\quad \times \exp\left(-\frac{1}{2} \sum_j (\sum_k A^{-1}_{jk}(x_k - \mu_k))(\sum_m A^{-1}_{jm}(x_m - \mu_m)) \right) dx_1 \cdots dx_d
\end{aligned}$$

Let us concentrate on the exponent:

$$\begin{aligned}
-\frac{1}{2} \sum_j (\sum_k A^{-1}_{jk}(x_k - \mu_k))(\sum_m A^{-1}_{jm}(x_m - \mu_m)) &= -\frac{1}{2} \sum_{k,m} (x_k - \mu_k) \sum_j (A^{-1}_{kj})^\top A^{-1}_{jm}(x_m - \mu_m) = \\
&= -\frac{1}{2} \sum_{k,m} (x_k - \mu_k) \Sigma^{-1}_{km} (x_m - \mu_m) = \\
&= -\frac{1}{2} (x - \mu) \Sigma^{-1} (x - \mu)^\top
\end{aligned}$$

where we used that $\Sigma^{-1} = (AA^\top)^{-1} = (A^\top)^{-1}A^{-1} = (A^{-1})^\top A^{-1}$.
Moreover, note that $\det \Sigma = \det A \det A^\top = (\det A)^2$ and that $\det A^{-1} = (\det A)^{-1}$.
So we get

$$E_P\Big(g(X)\Big) = \int_{\mathbb{R}^d} g(x_1, \dots, x_d) p_X(x_1, \dots, x_d) dx_1 \dots dx_d.$$

3. Let $(X, Y) \sim \mathcal{N}(0, \Sigma)$ jointly Gaussian random vectors with $X(\omega) \in \mathbb{R}^{n_x}$ and $Y(\omega) \in R^{n_y}$, with means $E(X) = \mu_X$ $E(Y) = \mu_Y$, and covariance

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^\top & \Sigma_{yy} \end{pmatrix}$$

Use Bayes formula to compute the conditional densities

$$p_{X|Y}(x|Y = y) \quad \text{and} \quad p_{Y|X}(y|X = x)$$

Solution

We obtain all the identities by using the formula multivariate gaussian formula. Consider $D = \Sigma^{-1}$, $X(\omega) \in R^{n_x}, Y(\omega) \in R^{n_y}$, $n = n_x + n_y$. Σ is the covariance matrix of (X, Y).
We define the notation $|A| := |\det A|$, A being a matrix. For the sake of simplicity, let be $\mu_X = \mu_Y = 0$. Later we will consider the general case with non vanishing expectations.
By Bayes' formula we have

$$p_{XY}(x, y) = (2\pi)^{-n/2}\sqrt{|D|} \exp\left(-\frac{1}{2}\Big\{(x, y)D(x, y)^\top\Big\}\right) = p_X(x)p_{Y|X}(y|x)$$

where

$$p_X(x) = (2\pi)^{-n_x/2}|\Sigma_{xx}|^{-1/2} \exp\left(-\frac{1}{2}\Big\{x\Sigma_{xx}^{-1}x^\top\Big\}\right)$$

$$\begin{aligned}
p_{Y|X}(y|x) = \frac{p_{XY}(x, y)}{p_X(x)} &= (2\pi)^{-n_y/2}\sqrt{|D| \times |\Sigma_{xx}|} \exp\left(-\frac{1}{2}\Big\{(x, y)D(x, y)^\top - x\Sigma_{xx}^{-1}x^\top\Big\}\right) = \\
(2\pi)^{-n_y/2}\sqrt{|D| \times |\Sigma_{xx}|} &\exp\left(-\frac{1}{2}\Big\{x(D_{xx} - \Sigma_{xx}^{-1})x^\top\right) \exp\left(-\frac{1}{2}\Big\{yD_{yy}y^\top + 2yD_{yx}^\top x^\top\Big\}\right) = \\
(2\pi)^{-n_y/2}\sqrt{|D| \times |\Sigma_{xx}|} &\exp\left(-\frac{1}{2}\Big\{x(D_{xx} - D_{xy}D_{yy}^{-1}D_{xy}^\top - \Sigma_{xx}^{-1})x^\top\Big\}\right) \\
\times \exp\left(-\frac{1}{2}\Big\{(y + xD_{xy}D_{yy}^{-1})D_{yy}(y + xD_{xy}D_{yy}^{-1})^\top\Big\}\right)
\end{aligned}$$

where in the last line we used the trick of the completion of the square.
Now conditionally on X we treat x as a constant. It follows that the conditional distribution $p_{Y|X}(y|x)$ is gaussian and the normalization constraint implies that the conditional covariance matrix is

$$\Sigma_{y|x} = D_{yy}^{-1}$$

and conditional mean

$$E(y|x) = -xD_{xy}D_{yy}^{-1}$$

Also since this conditional variance does not depend on x we must have

$$\Sigma_{xx}^{-1} = D_{xx} - D_{xy}D_{yy}^{-1}D_{xy}^\top$$

and also

$$|D_{yy}| = |D| \times |\Sigma_{xx}| = |\Sigma_{xx}|/|\Sigma|$$

Note also that by inverting the roles of Σ and D ($D = \Sigma^{-1}$ is also a symmetric non-negative matrix, which corresponds to a covariance matrix), we obtain

$$D_{xx}^{-1} = \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{xy}^{\top}, \tag{1}$$
$$|\Sigma_{yy}| = |\Sigma| \times |D_{xx}| = |D_{xx}|/|D| \tag{2}$$

By changing the roles of x and y we obtain also

$$\Sigma_{x|y} = D_{xx}^{-1}, \tag{3}$$
$$\Sigma_{yy}^{-1} = D_{yy} - D_{xy}^{\top}D_{xx}^{-1}D_{xy}, \tag{4}$$
$$D_{yy}^{-1} = \Sigma_{yy} - \Sigma_{xy}^{\top}\Sigma_{xx}^{-1}\Sigma_{xy} = \Sigma_{y|x} \tag{5}$$

Now we use the property of the inverse matrix: since $\Sigma D = D\Sigma = Id$

$$\begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^{\top} & \Sigma_{yy} \end{pmatrix} \begin{pmatrix} D_{xx} & D_{xy} \\ D_{xy}^{\top} & D_{yy} \end{pmatrix} = \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix} = \begin{pmatrix} D_{xx} & D_{xy} \\ D_{xy}^{\top} & D_{yy} \end{pmatrix} \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^{\top} & \Sigma_{yy} \end{pmatrix}$$

we have

$$\begin{aligned}
&\Sigma_{xx}D_{xx} + \Sigma_{xy}D_{xy}^{\top} = Id = D_{xx}\Sigma_{xx} + D_{xy}\Sigma_{xy}^{\top} \\
&\Sigma_{xx}D_{xy} + \Sigma_{xy}D_{yy} = 0 = D_{xx}\Sigma_{xy} + D_{xy}\Sigma_{yy} \\
&\Sigma_{xy}^{\top}D_{xx} + \Sigma_{yy}D_{xy}^{\top}0 = D_{xy}^{\top}\Sigma_{xx} + D_{yy}\Sigma_{xy}^{\top} \\
&\Sigma_{xy}^{\top}D_{xy} + \Sigma_{yy}D_{yy}^{\top} = Id = D_{xy}^{\top}\Sigma_{xy} + D_{yy}\Sigma_{yy}^{\top}
\end{aligned}$$

We can use it to obtain a new expression for the conditional expectation:

$$\begin{aligned}
&E(y|x) = -xD_{xy}D_{yy}^{-1} = -xD_{xy}\big(\Sigma_{yy} - \Sigma_{xy}^{\top}\Sigma_{xx}^{-1}\Sigma_{xy}\big) \\
&= x\big(-D_{xy}\Sigma_{yy} + D_{xy}\Sigma_{xy}^{\top}\Sigma_{xx}^{-1}\Sigma_{xy}\big) \\
&= x\big(D_{xx}\Sigma_{xy} + \{Id - D_{xx}\Sigma_{xx}\}\Sigma_{xx}^{-1}\Sigma_{xy}\big) \\
&= x\big(D_{xx}\Sigma_{xy} + \Sigma_{xx}^{-1}\Sigma_{xy} - D_{xx}\Sigma_{xy}\big) = x\Sigma_{xx}^{-1}\Sigma_{xy}
\end{aligned}$$

By changing the roles of x and y we get also

$$E(x|y) = -yD_{xy}^{\top}D_{xx}^{-1} = y\Sigma_{yy}^{-1}\Sigma_{xy}^{\top}$$

When X and Y a priori have non zero mean, by using $X' = (X - \mu_X)$ and $Y' = (Y - \mu_Y)$ we obtain

$$E(X|Y) = \mu_X + (Y - \mu_Y)\Sigma_{yy}^{-1}\Sigma_{xy}^{\top} \tag{6}$$

$$E(Y|X) = \mu_Y + (X - \mu_X)\Sigma_{xx}^{-1}\Sigma_{xy} \tag{7}$$

It follows also that

$$D_{xy} = -\Sigma_{xx}^{-1}\Sigma_{xy}D_{yy} = -\Sigma_{xx}^{-1}\Sigma_{xy}\Sigma_{y|x}^{-1} = -\Sigma_{x|y}^{-1}\Sigma_{xy}\Sigma_{yy}^{-1}$$

and

$$D = \begin{pmatrix} D_{xx} & D_{xy} \\ D_{xy}^\top & D_{yy} \end{pmatrix} = \Sigma^{-1} = \begin{pmatrix} \Sigma_{x|y}^{-1} & -\Sigma_{x|y}^{-1}\Sigma_{xy}\Sigma_{yy}^{-1} \\ -\Sigma_{xx}^{-1}\Sigma_{xy}\Sigma_{y|x}^{-1} & \Sigma_{y|x}^{-1} \end{pmatrix}$$

To sum up, we obtained that both $p_{X|Y}(x|Y=y)$ and $p_{Y|X}(y|X=x)$ are Gaussian distribution and their explicit formulae read

$$\begin{aligned} p_{X|Y}(x|Y=y) &= (2\pi)^{-n_x/2}|D_{xx}|^{1/2}\exp\left(-\frac{1}{2}\left\{[x-E(x|y)]\,D_{xx}\,[x-E(x|y)]^\top\right\}\right) \\ p_{Y|X}(y|X=x) &= (2\pi)^{-n_y/2}|D_{yy}|^{1/2}\exp\left(-\frac{1}{2}\left\{[y-E(y|x)]\,D_{yy}\,[y-E(y|x)]^\top\right\}\right) \end{aligned}$$

where $E(y|x)$ and $E(x|y)$ are given by 6 and 7 and D_{xx}^{-1} and D_{yy}^{-1} are given by 1 and 5.

4. Consider a random variable $Y(\omega) \in L^2(\Omega, \mathcal{F}, P)$. Consider the linear subspace spanned by the random variable $Y(\omega)$.

$$\begin{aligned} &\text{LinearSpan}(Y)\{b + aY(\omega) : a, b \in \mathbb{R}\} \\ &\subset L^2(\Omega, \sigma(Y), P) = \{g(Y(\omega)) :\ g(y) \text{ Borel measurable }\} \cap L^2(\Omega, \mathcal{F}, P) \end{aligned}$$

(a) Show that $LinearSpan(Y)$ is a closed subspace of $L^2(\Omega, \mathcal{F}, P)$.

(b) Let X a random variable in $L^2(\Omega, \mathcal{F}, P)$. Compute the orthogonal projection of X on $LinearSpan(Y)$.
Hint: you can assume that $E(X) = 0$ and $E(Y) = 0$.

Solution

(a) Consider a sequence $Z_n \in LinearSpan(Y)$ and its limit $Z \in L^2(\Omega, \mathcal{F}, P)$. Note that in fact $LinearSpan(Y) = Span\{1, Y\} = Span\{1, Y/E(Y^2)\}$, so the orthonormal projection of Z on $LinearSpan(Y)$ is simply

$$\Pi_{LinearSpan(Y)}(Z) = E(Z) + E(ZY)Y/E(Y^2) \in LinearSpan(Y)$$

Now consider di L^2-distance between Z_n and $\Pi_{LinearSpan(Y)}(Z)$:

$$\begin{aligned} E(Z_n - \Pi_{LinearSpan(Y)}(Z))^2 &= E\left(Z_n - E(Z) - \frac{E(ZY)}{E(Y^2)}Y\right)^2 = \\ &= E\left(E(Z_n) + \frac{E(Z_nY)}{E(Y^2)}Y - E(Z) - \frac{E(ZY)}{E(Y^2)}Y\right)^2 \\ &= E\left[E(Z_n - Z) + Y\frac{E((Z_n - Z)Y)}{E(Y^2)}\right]^2 \\ &= (E(Z_n - Z))^2 + \frac{(E((Z_n - Z)Y))^2}{E(Y^2)} \\ &\leq 2E(Z_n - Z)^2 \to 0 \end{aligned}$$

where we used the Schwartz inequality.
Thus, we have shown that $Z_n \to \Pi_{LinearSpan(Y)}(Z)$ and, since the limit is unique, we have necessarily that $Z = \Pi_{LinearSpan(Y)}(Z) \in LinearSpan(Y)$.

(b) The orthogonal projection of X is

$$\Pi_{LinearSpan(Y)}(X) = E(X) + \frac{E(XY)}{E(Y^2)}Y = \frac{E(XY)}{E(Y^2)}Y$$

5. Consider a jointly Gaussian pair of random variables (X, Y), with means $E(X) = 0$ and $E(Y) = 0$, and covariance

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix}$$

(a) Compute the orthogonal projection of X on $LinearSpan(Y)$ and show that it coincides with on $E(X|Y)(\omega)$.
Hint: compute the orthogonal projection by minimizing w.r.t. a, b

$$E\big((b + aY - X)^2\big)$$

(b) Compute the conditional variance of X given Y, defined as

$$\text{Cov}(X|Y)(\omega) = E(X^2|Y)(\omega) - E(X|Y)(\omega)^2$$

Solution

(a) By exercise 3 we now that $E(X|Y) = Y\Sigma_{XY}\Sigma_{YY}^{-1}$ (note that in this case Σ_{XY} and Σ_{YY} are just numbers and $\Sigma_{XY} = \Sigma_{YX}$).
On the other hand we compute the orthogonal projection of X by minimizing with respect to a, b

$$E\big((b + aY - X)^2\big) = b^2 + a^2E(Y^2) + E(X^2) - 2aE(XY)$$

We look for the stationary points:

$$\begin{aligned} \partial_b E\big((b + aY - X)^2\big) &= 2b = 0 \\ \partial_b E\big((b + aY - X)^2\big) &= 2aE(Y^2) - 2E(XY) = 0 \end{aligned}$$

then, we get that the stationary point is $b = 0$ and $a = E(XY)/E(Y^2) = \Sigma_{XY}/\Sigma_{YY}$. A simple calculation shows that the Hessian matrix at the stationary point is $H = diag(2, 2E(Y^2))$ which is positive definite, so our stationary point is a minimum point. This means that

$$\Pi_{LinearSpan(Y)}(X) = Y\Sigma_{XY}\Sigma_{YY}^{-1} = E(X|Y).$$

(b) Again by exercise 3 we readily get that

$$\text{Cov}(X|Y)(\omega) = E(X^2|Y)(\omega) - E(X|Y)(\omega)^2 = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$$

6. Let $0 < s < t < u$, and $(B_r : r \geq 0)$ a standard Brownian motion with $B_0 = 0$. Compute the conditional distribution of B_u conditionally on $\sigma(B_s, B_u)$.

Solution
Let be $X = B_t$ and $Y = (B_s \ \ B_t)^\top$. Of course we have $EX = 0$ and $EY = (0 \ \ 0)^\top$. Then we have

$$\begin{aligned} \Sigma_{XX} &= t \\ \Sigma_{XY} &= (E(B_tB_s) \ \ E(B_tB_u) = (s \ \ t) \\ \Sigma_{YY} &= \begin{pmatrix} E(B_sB_s) & E(B_sB_u) \\ E(B_uB_s) & E(B_uB_u) \end{pmatrix} = \begin{pmatrix} s & s \\ s & u \end{pmatrix} \end{aligned}$$

Now, by using the result in exercise 3, we obtain that the distribution of the random variable $(B_t|B_s = b_1, B_u = b_2)$ is Gaussian with expectation value

$$\mu = \Sigma_{XY}\Sigma_{YY}^{-1}Y = \frac{u-t}{u-s}b_1 + \frac{t-s}{u-s}b_1$$

and variance

$$\Sigma = \Sigma_{XX} + \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX} = \frac{(u-t)(t-s)}{u-s}$$

7. A d-dimensional Brownian motion is an $\mathbb{R}^d$-valued stochastic process $B_t = (B_t^{(1)}, \dots, B_t^{(d)})$, $t \geq 0$, where the components $B_t^{(k)}$ are independent $\mathbb{R}$-valued standard Brownian motions.

   (a) Let Q be an orthogonal $d \times d$-matrix , which means $QQ^\top = Q^\top Q = Id$, (equivalently $Q^{-1} = Q^\top$). Show that $(QB_t : t \geq 0)$ is a d-dimensional Brownian motion.

   (b) For a matrix $A \in d \times d$, let $X_t = (AB_t) \in \mathbb{R}^n$. Show that $(X_t : t \geq 0)$ is a Gaussian process (all finite dimensional distributions are jointly Gaussian), with independent jointly Gaussian increments, i.e. for $0 \leq s \leq t$, $(X_t - X_s) \perp\!\!\!\perp \mathcal{F}_s^X = \sigma(X_r : r \leq s)$.

   (c) compute the covariance $E(X_t^{(i)} X_s^{(j)})$. Compute the stochastic cross variations

$$[X^{(i)}, X^{(j)}]_t = \lim_{n\to\infty} \sum_{t_k^n \in \Pi^n} \big(X^{(i)}_{t_k^n \wedge t} - X^{(i)}_{t_{k-1}^n \wedge t}\big)\big(X^{(j)}_{t_k^n \wedge t} - X^{(j)}_{t_{k-1}^n \wedge t}\big)$$

   for any sequence of partitions (Π^n) with $\Delta(\Pi^n, t) \to 0$, where we take limit in probability and the limit does not depend on the particular sequence (Π^n), and we have also P-almost sure convergence when $\sum_{n\in\mathbb{N}} \Delta(\Pi^n, t) < \infty$.

Solution

(a) First, note that $W_t^{(j)} := \sum_k Q_{jk} B_t^{(k)}$ is still Gaussian with vanishing mean since is a linear combination of Gaussian random variables with vanishing mean and is independent from $W_s^{(j')}$ for $j \neq j'$ since

$$\begin{aligned} E(W_t^{(j)} W_s^{(j')}) &= E\left(\sum_{k,k'} Q_{jk} Q_{j'k'} B_t^{(k)} B_s^{(k')}\right) = \\ &= \sum_{k,k'} Q_{jk} Q_{j'k'} E\left(B_t^{(k)} B_s^{(k')}\right) = \\ &= (t \wedge s) \sum_{k,k'} Q_{jk} Q_{j'k'} \delta_{k,k'} = (t \wedge s)\delta_{jj'} \end{aligned}$$

The last equation means also that $E(W_j(t)W_j(s)) = t \wedge s$ as we expect.
Of course $W_t^{(j)}$ is also continuous almost surely as linear combination of almost surely continuous processes.
Now we need to show the independence of the increments: in fact, let $1 < t_2 < t_3 < t_4$ and consider

$$E(W_j(t_4) - W_j(t_3))(W_j(t_2) - W_j(t_1)) = (t_4 \wedge t_3) - (t_3 \wedge t_2) - (t_4 \wedge t_1) + (t_3 \wedge t_1) = 0$$

(b) We can see that each component $X_t^{(i)}$ is a Gaussian process by looking at the distribution of the vector $(X_{t_1}^{(j_1)}, \dots, X_{t_n}^{(j_n)})$ where $n \in \mathbb{N}$ is arbitrary. In particular we want to show that this distribution is jointly Gaussian.
Consider the characteristic function

$$\begin{aligned} E\left(\exp\left\{i \sum_{k=1}^n \lambda_k X_{t_k}^{(j_k)}\right\}\right) &= E\left(\exp\left\{i \sum_{k=1}^n \lambda_k \sum_{m=1}^d A_{j_k m} B_{t_k}^{(m)}\right\}\right) = \\ &= \prod_{m=1}^d E\left(\exp\left\{i \sum_{k=1}^n \lambda_k A_{j_k m} B_{t_k}^{(m)}\right\}\right) \end{aligned}$$

Now we can define $\theta_{k,m} := \lambda_k A_{j_k m}$ so that

$$\begin{aligned}
E\left(\exp\left\{i\sum_{k=1}^{n}\lambda_k X_{t_k}^{(j_k)}\right\}\right) &= \prod_{m=1}^{d} E\left(\exp\left\{i\sum_{k=1}^{n}\theta_{k,m}B_{t_k}^{(m)}\right\}\right) = \\
&= \prod_{m=1}^{d}\exp\left\{-\frac{1}{2}\sum_{k,k'}^{n}\theta_{k,m}\theta_{k',m}(t_k\wedge t_{k'})\right\} = \\
&= \exp\left\{-\frac{1}{2}\sum_{k,k'}^{n}\sum_{m=1}^{d}\lambda_k\lambda_{k'}A_{j_k m}A_{j_{k'} m}(t_k\wedge t_{k'})\right\} = \\
&= \exp\left\{-\frac{1}{2}\sum_{k,k'}^{n}\lambda_k\lambda_{k'}\Sigma_{j_k j_{k'}}(t_k\wedge t_{k'})\right\}
\end{aligned}$$

where $\Sigma = AA^\top$ is a $d\times d$ matrix. Therefore, we obtained that the process is a Gaussian process with vanishing expectation and covariance

$$Cov(X_{t_1}^{j_1}X_{t_2}^{j_2}) = \Sigma_{j_1 j_2}(t_1\wedge t_2)$$

Now, in order to show that the increments are independent, for $t_1 < t_2 < t_3 < t_4$, just consider

$$E(X_{t_4}^{(i)} - X_{t_3}^{(i)})(X_{t_2}^{(j)} - X_{t_1}^{(j)}) = \Sigma_{ij}\left[(t_4\wedge t_2) - (t_3\wedge t_2) - (t_4\wedge t_1) + (t_3\wedge t_1)\right] = 0$$

(c) We have already computed the covariance, so we just need to calculate the stochastic cross variation:

$$\begin{aligned}
[X^{(i)}, X^{(j)}]_t &= \lim_{n\to\infty}\sum_{t_k^n\in\Pi^n}\big(X_{t_k^n\wedge t}^{(i)} - X_{t_{k-1}^n\wedge t}^{(i)}\big)\big(X_{t_k^n\wedge t}^{(j)} - X_{t_{k-1}^n\wedge t}^{(j)}\big) = \\
&= \lim_{n\to\infty}\sum_{t_k^n\in\Pi^n}\sum_{m,m'}A_{im}A_{jm'}\big(B_{t_k^n\wedge t}^{(m)} - B_{t_{k-1}^n\wedge t}^{(m)}\big)\big(B_{t_k^n\wedge t}^{(m')} - B_{t_{k-1}^n\wedge t}^{(m')}\big) = \\
&= \sum_{m,m'}A_{im}A_{jm'}\lim_{n\to\infty}\sum_{t_k^n\in\Pi^n}\big(B_{t_k^n\wedge t}^{(m)} - B_{t_{k-1}^n\wedge t}^{(m)}\big)\big(B_{t_k^n\wedge t}^{(m')} - B_{t_{k-1}^n\wedge t}^{(m')}\big) = \\
&= \sum_{m,m'}A_{im}A_{jm'}\delta_{m,m'}t = t\Sigma_{ij}
\end{aligned}$$