Stochastic analysis, Fall 2014, Exercise 9,3.11.2013

1. (Bougerol' identity) Consider

   $\sinh(W_t)$ where W_t is Brownian motion and

   $$\sinh(x) = \frac{e^x - e^{-x}}{2}, \ \cosh(x) = \frac{e^x + e^{-x}}{2},$$
   $$\frac{d}{dx}\sinh(x) = \cosh(x), \ \frac{d}{dx}\cosh(x) = \sinh(x), \ \cosh(x)^2 - \sinh(X)^2 = 1$$

   Let also B_t and β_t independent Brownian motions, and

   $$X_t = e^{B_t} \int_0^t e^{-B_s} d\beta_s \tag{1}$$

   Apply Ito's formula to show that

   the processes $(X_t : t \geq 0)$ and $(\sinh(W_t) : t \geq 0)$ have the same distributions,

   i.e. they satisfy the same stochastic differential equation in Ito sense.

2. We consider a $\mathbb{C}$-valued continuous local martingale

   $$Z_t = X_t + iY_t$$

   where X_t and Y_t are $\mathbb{R}$-valued continuous local martingales in a filtration $\mathbb{F}$.

   (a) Prove that there is an unique continuous $\mathbb{C}$-valued process of finite variation $\langle Z, Z\rangle_t$ such that

   $$Z^2 - \langle Z, Z\rangle_t$$

   is a complex local martingale.

   (b) Prove that the following statements are equivalent:

   i. Z_t^2 is a $\mathbb{F}$-local martingale.

   ii. $\langle Z, Z\rangle_t = 0 \ \forall t$

   iii. $\langle X, X\rangle = \langle Y, Y\rangle$ and $\langle X, Y\rangle = 0$

   Such $\mathbb{C}$-valued local martingales are called *conformal* local martingales. For example, if B_t and β_t are independent $\mathbb{R}$-valued Brownian motions in the filtration $\mathbb{F}$, the $\mathbb{C}$-valued *planar Brownian motion*

   $$W_t = B_t + i\beta_t \tag{2}$$

   is a conformal martingale.

   (c) Use Lévy characterization theorem to show that if $Z_t = X_t + iY_t$ is a C-valued continuous conformal local martingale in a filtration F, there exists a $\mathbb{C}$-valued planar Brownian motion, such that

   $$Z_t = W_{\langle X,X\rangle_t}$$

   with

   $$W_u = X_{\tau(u)} + iY_{\tau(u)}, \quad \tau(u) = \inf\{t : \langle X, X\rangle_t \geq u\}$$

(d) Show that for a conformal local martingale $Z_t = X_t + iY_t$

$$\langle X, X\rangle_t = \langle \Re(Z), \Re(Z)\rangle_t = \frac{1}{2}\langle Z, \overline{Z}\rangle_t \tag{3}$$

(e) Let H_t a $\mathbb{C}$-valued bounded progressive process, and Z_t a conformal local martingale, Then

$$U_t = \int_0^t H_s dZ_s = \int_0^t \Re(H_s)dX_s - \int_0^t \Im(H_s)dY_s + i\left(\int_0^t \Im(H_s)dX_t + \int_0^t \Re(H_s)dY_s\right)$$

is a conformal martingale with

$$\langle U, \overline{U}\rangle_t = \int_0^t |H_s| d\langle Z, \overline{Z}\rangle_t$$

(f) Let Z_t a continuous conformal local martingale and $F : \mathbb{C} \to \mathbb{C}$ twice differentiable as function of two real variables, use Ito formula to show

$$F(Z_t) = F(Z_0) + \int_0^t \frac{\partial}{\partial z}(Z_s)dZ_s + \int_0^t \frac{\partial}{\partial \overline{z}}(Z_s)d\overline{Z}_s + \frac{1}{4}\int_0^t \Delta F(Z_s)d\langle Z, \overline{Z}\rangle_s$$

where for $z = x + iy$

$$\Delta F(z) = \Delta(F(x,y)) = \frac{\partial^2}{\partial x^2}F(x,y) + \frac{\partial^2}{\partial y^2}F(x,y)$$
$$= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)F(x,y) = \frac{\partial^2}{\partial z \partial \overline{z}}F(x,y)$$

(g) We say that F is harmonic if $\Delta F(z) = 0 \ \forall z \in \mathbb{C}$. Show that if F is harmonic and Z_t a continuous conformal local martingale, than $F(Z_t)$ is a local martingale.

(h) Let

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

A function $F : \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if

$$\frac{\partial}{\partial \overline{z}}F(z) = 0$$

and in such case we set

$$F'(x) = \frac{\partial}{\partial z}F(z) \ .$$

If $F(z)$ is holomorphic it is necessarily harmonic. Show that in such case, if Z_t a continuous conformal local martingale, then $F(Z_t)$ is a continuous conformal local martingale with

$$F(Z_t) = F(Z_0) + \int_0^t F'(Z_s)dZ_s$$

(i) Show that if W_t is a $\mathbb{C}$-valued planar Brownian motion and $F : \mathbb{C} \to \mathbb{C}$ is holomorphic and non-constant

$$F(W_t) = F(W_0) + \widetilde{W}_{\langle X,X\rangle_t}$$

where $X_t = \Re(F(W_t))$ and

$$\langle X, X\rangle_t = \int_0^t |F'(W_s)|^2 ds$$

and $\widetilde{W_t}$ is a $\mathbb{C}$-valued planar Brownian motion.
In other words $F(W_t)$ is a time changed Brownian motion. We say that Brownian motion is *conformal invariant* .
Note We can also show that $\langle X, X\rangle_\infty = \infty$.