Stochastic analysis, fall 2014, Exercises-8, 26.11.2013

It is not true that all uniformly integrable local martingales are true martingales. Even local martingales bounded in $L^2(P)$ need not to be true martingales ! Here we study such counterexample.

Let $B_t = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)})$ a 3-dimensional brownian motion starting from 0 at time 0 , with independent components, so that $\langle B^{(i)}, B^{(j)} \rangle_t = \delta_{ij}$.

The process

$$R_t = |B_t| = \sqrt{\sum_{i=1}^{3} \big(B_s^{(i)}\big)^2}$$

is called the 3-dimensional Bessel process.

1. Use Ito formula to compute the semimartingale decomposition of R_t into a continuous local martingale part W_t and a continuous process of finite variation.
2. Compute $\langle R \rangle_t = \langle W \rangle_t$ and use Paul Lévy's characterization theorem for Brownian motion to show that the local martingale part of R_t which satisifies

   $$W_t = R_t - \int_0^s \frac{1}{R_s} ds$$

   is a Brownian motion in the filtration $\mathbb{F}$ generated by (B_t).
3. Show that R_t is a $\mathbb{F}$-submartingale.

   Let $M_t = R_t^{-1}$ for $t \geq 1$. We start the process at time 1 since $R_0 = 0$.
4. Use Ito formula to show that $(M_t)_{t \geq 1}$ is a local martingale, and write its Ito integral representation.
5. Compute $\langle M \rangle_t$.
6. Show that (M_t) is a supermartingale. Hint: it is non-negative, you can use Fatou lemma for the sequence of localized martingales $(M_{t \wedge \tau_n} : t \geq 1)$, $n \in \mathbb{N}$.
7. Let $\tau_a := \inf\{t \geq 1 : R_t = a\}$, $a > 0$, with the convention $\inf\{\emptyset\} = \infty$.

   Show that the stopped process $(M_t^{\tau_a})_{t \geq 1}$ is a martingale and consequently $(\tau_{1/n} : n \in \mathbb{N})$

   is a localizing sequence for the local martingale $(M_t : t \geq 1)$
8. Let $0 < r' < y < r''$. Use the martingale property of $(M_{t \wedge \tau_r} : t \geq 1)$ to compute $P(\tau_{r'} < \tau_{r''} | R_1 = y)$. By the conditioning we mean that we start R_t at time $t = 1$ in position y.
9. For $0 < r < y$ compute also $P(\tau_r < \infty | R_1 = y)$.

10. Using (g), show that the 3-dimensional Brownian motion is transient, $|B_t| \to \infty$ P a.s., meaning that it leaves eventually any ball centered around the origin withou coming back, and therefore $M_\infty = \lim_{t\to\infty} M_t = 0$.

11. Using the multivariate gaussian density in polar coordinates, compute the probability densities of R_t and M_t, and show that the local martingale $(M_t : t \geq 1)$ is bounded in L^2 , so that in particular it is uniformly integrable. (We start the martingale at time 1 since there is the square norm explodes near the origin).

    Note however that $(M_t)_{t\geq 1}$ is not a martingale. Otherwise (X_t) would be an uniformly integrable martingale so that $M_t = E(M_\infty|\mathcal{F}_t)$, $t \geq 1$. But in dimension 3 the Brownian motion is transient, which means that $M_\infty = 0$.

12. Compute also the probability distribution of R_t^2.

    Show first that for $t = 1$,

$$P(R_1^2 \in dx) = \mathbf{1}(x \geq 0)\frac{1}{\Gamma(3/2)2^{3/2}} \exp(-x/2)x^{\frac{3}{2}-1}dx$$

    which is the distribtion of a Gamma random variable with shape parameter 3/2 and scale parameter 2, (also called chi-square with 3 degrees of freedom and use the scaling property of Brownian motion.

13. Show that $E(\langle M\rangle_t) = \infty$ $\forall t \geq 1$.

    Remark This is not in contradiction with $E(M_\infty^2) < \infty$, since

$$E\big((M_t - M_1)^2\big) = E\big(\langle M\rangle_t - \langle M\rangle_1\big)$$

    holds for martingales but does not need to hold for local martingales. Even if the local martingale M_t is bounded in L^2, it means that M_t^2 is bounded in L^1 which does not give the uniform integrability condition which is necessary to take the limit of a localizing sequence under the expectation.

Remark In general, when M_t is a continuous local martingale with localizing sequence $\tau_n \uparrow \infty$, to show that it is a true martingale, you need to show that for $s \leq t$ and $A \in \mathcal{F}_s$

$$E_P(M_{t\wedge\tau_n}\mathbf{1}_A) \overset{?}{\to} E_P(M_t\mathbf{1}_A)$$
$$E_P(M_{s\wedge\tau_n}\mathbf{1}_A) \overset{?}{\to} E_P(M_s\mathbf{1}_A)$$

where the left sides are equal by since $(M_{t\wedge\tau_n} : t \geq 0)$ is a martingale. When the local martingale $(M_t : t \geq 0)$ is uniformly integrable or bounded in $L^2(P)$, for fixed t, the sequence $(M_{t\wedge\tau_n} : n \in \mathbb{N})$ does not need to be uniformly integrable, which is the condition that we need to take the limit inside the expectation.

Of course if the local martingale is bounded on finite intervals, $|M_t(\omega)| \leq c(t) < \infty$ P a.s., then it is a true martingale.

Also, when $M_t \in L^2(P)$ is a martingale and $\forall t$ and Y_t is a progressive integrand with

$$E\left(\int_0^t Y_s^2 d\langle M\rangle_s\right) < \infty \quad \forall t \tag{1}$$

then the Ito integral $(Y \cdot M)_t \in L^2(P)$ is a true martingale.

If M_t is not a square integrable martingale, or Y does not satisfy the condition 1, the stochastic integral $(Y \cdot M)_t$ is just local martingale.