**Stochastic analysis, fall 2014, Exercises-7, 12.11.2014**

1. Let $\tau(\omega) \in [0, +\infty]$ be a random time, $F(t) = P(\tau \le t)$ for $t \in [0, \infty)$.

   Consider the single jump *counting process* $N_t(\omega) := \mathbf{1}(\tau(\omega) \le t)$ which generates the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $\mathcal{F}_t^N = \sigma(N_s : s \le t)$.

   (a) Show that $\tau$ is a stopping time in the filtration $\mathbb{F}$.

   (b) Show that first that for every Borel function $f(x)$, the random variable

   $$f(\tau(\omega))\mathbf{1}(\tau(\omega) \le s)$$

   is $\mathcal{F}_s$-measurable.

   (c) Define the *cumulative hazard function*

   $$\Lambda(t) = \int_0^t \frac{1}{1 - F(s-)} F(ds)$$

   where $F(s-) = P(\tau < s)$ denotes the limit from the left.
   Show that

   $$M_t = N_t - \Lambda_{t \wedge \tau}$$

   is a an $\mathbb{F}$-martingale.
   Hint: use the definition, and show that for $s \le t$ and every $A \in \mathcal{F}_s$

   $$E_P\Big( (N_t - N_s)\mathbf{1}_A \Big) = E_P\Big( (\Lambda_{t\wedge\tau} - \Lambda_{s\wedge\tau})\mathbf{1}_A \Big)$$

   It turns out that it is enough to do the computation for $A = \{\omega : \tau(\omega) > s\}$ (why ?). Fubini's theorem may be also useful.

   (d) Assume that $t \mapsto F(t)$ and therefore also $t \mapsto \Lambda(t)$ are continuous, which means $P(\tau = t) = 0 \ \forall t \in \mathbb{R}^+$.
   Show that $\Lambda_\tau$ has 1-exponential distribution:

   $$P\big(\Lambda_\tau > x\big) = \exp(-x), \quad x \ge 0$$

   Hint: one line of proof compute the Laplace transform

   $$\mathcal{L}(\theta) := E_P\Big( \exp(-\theta \Lambda_\tau) \Big) \quad \theta > 0$$

   and compare it with the Laplace transform of the 1-exponential distribution.

   (e) Show that the martingale $M_t$ is uniformly integrable, what is $M_\infty$?.

2. Let $(M_t : t \in \mathbb{R}^+)$ a $\mathbb{F}$-martingale, and $\mathbb{G}$ a filtration with $\mathcal{G}_t \subseteq \mathcal{F}_t$ We assume that $(M_t)$ is also $\mathbb{G}$-adapted. Show that $(M_t)$ is a martingale in the smaller filtration $\mathbb{G}$.

3. Let $(M_t : t \in \mathbb{R})$ a $F$-martingale under $P$, and $\mathcal{G}_t$ a filtration such that $\forall t \geq 0$, the $\sigma$-algebrae $\mathcal{G}_t$ and $\sigma(M_s : s \leq t)$ are $P$-independent.

   Show that under $P$, $(M_t : t \in \mathbb{R}^+)$ is a martingale in the enlarged filtration $(\mathcal{F}_t \vee \mathcal{G}_t : t \geq 0)$.

4. Let $(B_t : t \geq 0)$ a Brownian motion in the filtration $\mathbb{F}$, which means

   - $B_0(\omega) = 0$
   - $t \mapsto B_t(\omega)$ is continuous
   - $\forall 0 \leq s \leq t$ , $(B_t - B_s)$ is $P$-independent from $\mathcal{F}_s$, conditionally gaussian with conditional mean $E(B_t - B_s|\mathcal{F}_s) = 0$ and conditional variance $E((B_t - B_s)^2|\mathcal{F}_s) = t - s$

   Note: Since by definition an $\mathbb{F}$-Brownian motion $B$ is $\mathbb{F}$-adapted, the filtration $\mathbb{F}$ contains the filtration $\mathbb{F}^B(\mathcal{F}_t^B : t \geq 0)$ with $\mathcal{F}_t^B = \sigma(B_s : 0 \leq s \leq t)$, and it is possibly bigger.

5. Show that for $a > 0$ the process $(a^{-1/2}B_{at} : t \in \mathbb{R}^+)$ is also a Brownian motion.

6. The process $W_0 = 0$, $W_t = tB_{1/t}$ is also a Brownian motion.

7. Let $\theta \in \mathbb{R}$, and $i = \sqrt{-1}$ the imaginary unit

   Show that

   $$E_P\big(\exp(i\theta B_t)\big) = \exp\big(-\frac{1}{2}\theta^2 t\big)$$

   Hint: Use complex integration over the rectangular contour delimited by in the complex plane by the points $R, (R+i\theta), (-R+i\theta), -R$ with $R \in \mathbb{R}$ and let $R \to \infty$.

8. For $\theta \in \mathbb{R}$, consider now

   $$M_t = \exp\big(i\theta B_t + \frac{1}{2}\theta^2 t\big) = \Big\{\exp\big(\frac{1}{2}\theta^2 t\big)\cos(\theta B_t) + \sqrt{-1}\exp\big(\frac{1}{2}\theta^2 t\big)\sin(\theta B_t)\Big\} \in \mathbb{C}$$

   where $i = \sqrt{-1}$ is the imaginary unit.
   Recall that $E(\exp(i\theta G)) = \exp(-\theta^2\sigma^2/2)$ when $G(\omega) \sim \mathcal{N}(0, \sigma^2)$.

   - Show that $M_t$ is complex valued $\mathbb{F}$-martingale, which means that real and imaginary parts are $\mathbb{F}$-martingales.
   - Show that $\lim_{t\to\infty} |M_t(\omega)| = \infty$