Stochastic analysis, fall 2014, Exercises-6, 29.10.2014

Consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with the dicrete-time filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$

1. In discrete time, show that a $\mathbb{F}$-predictable $(P, \mathbb{F})$-martingale is constant, i,e $M_n(\omega) = M_0(\omega)\ \forall n$.

2. A *potential* $(Z_n : n \in \mathbb{N})$ is a non-negative $(P, \mathbb{F})$-supermartingale with

$$\lim_{n\to\infty} E(Z_n) = 0$$

   Show that a potential is uniformly integrable.

   Note: The potential terminology comes in analogy with physics, where potentials do vanish at infinity. Note that a potential is necessarly uniformly integrable.

3. An $(\mathbb{F}, P)$-supermartingale $(X_n : n \in \mathbb{N})$ has *Riesz decomposition* if it can be written as

$$X_n = Y_n + Z_n$$

   where Y_n is a martingale and Z_n is a potential.

   (a) Show that if $\sup_{n\in\mathbb{N}} E_P\big(X_n^-\big) < \infty$ then X_n has Riesz decomposition with

$$Y_n = M_n - E(A_\infty|\mathcal{F}_n), \quad Z_n = E(A_\infty|\mathcal{F}_n) - A_n,$$

   where $X_n = M_n - A_n$ is the Doob decomposition of X into a martingale part M and a predictable part with A non-decreasing and $A_0 = 0$.

   (b) Show that the Riesz decomposition is unique.

4. Show that a martingale $(X_t : t \in \mathbb{N})$ has *Krickeberg decomposition*

$$X_t = L_t - M_t$$

   where L_t and M_t are non-negative $(P, \mathbb{F})$-martingales , if and only if

$$\sup_{t\in\mathbb{N}} E_P\big(|X_t|\big) < \infty$$

   equivalently

$$\sup_{t\in\mathbb{N}} E_P\big(X_t^+\big) < \infty \text{ or } \sup_{t\in\mathbb{N}} E_P\big(X_t^-\big) < \infty$$

   Hints: You can always assume without loss of generality that $M_0 = 0$, otherwise consider the martingale $(M_t - M_0)$.

   For the necessity note that when M has Riesz decomposition, $|M_t| \le Y_t + M_t$.

For sufficency show take the decomposition $X_t = X_t^+ - X_t^-$, and show first that $(-X_t^-)$ is a a supermartingale and which admits Riesz decomposition

$$(-X_t^-) = Y_t + Z_t$$

where Y_t is a martingale and Z_t is a potential. Show then that X_t has Krickeberg decomposition with

$$L_t = (X_t - Y_t) = X_t^+ + Z_t \geq 0, \quad \text{and} \quad M_t = -Y_t = X_t^- + Z_t \geq 0.$$

5. Suppose we have an urn which contains at time $t = 0$ two balls, one black and one white. At each time $t \in N$ we draw uniformly at random from the urn one ball, and we put it back together with a new ball of the same colour.

   We introduce the random variables
   $X_t(\omega) = \mathbf{1}\{$ the ball drawn at time t is black $\}$
   and denote $S_t = (1 + X_1 + \cdots + X_t)$,
   $M_t = S_t/(t+2)$, the proportion of black balls in the urn.
   We use the filtration $\{\mathcal{F}_n\}$ with $\mathcal{F}_n = \sigma\{X_s : s \in \mathbb{N}, s \leq t\}$.

   i) Compute the Doob decomposition of (S_t), $S_t = S_0 + N_t + A_t$, where (N_t) is a martingale and (A_t) is predictable.

   ii) Show that (M_t) is a martingale and find the representation of (M_t) as a martingale transform $M_t = (C \cdot N)_t$, where (N_t) is the martingale part of (S_t) and (C_t) is predictable.

   iv) Note that the martingale $(M_t)_{t \geq 0}$ is uniformly integrable (Why ?). Show that P a.s. and in L^1 exists $M_\infty = \lim_{t \to \infty} M_t$. Compute $E(M_\infty)$.

   v) Show that $P(0 < M_\infty < 1) > 0$.

   Since $M_\infty(\omega) \in [0,1]$, it is enough to show that $0 < E(M_\infty^2) < E(M_\infty)$ with strict inequalities.

   Hint: compute the Doob decomposition of the submartingale (M_t^2), and than take expectations before going to the limit to find the value of $E(M_\infty^2)$.

6. Consider an i.i.d. random sequence $(U_t : t \in \mathbb{N})$ with uniform distribution on $[0,1]$, $P(U_1 \in dx) = \mathbf{1}_{[0,1]}(x)dx$. Note that $E_P(U_t) = 1/2$.

   Consider also the random variable $-\log(U_1(\omega))$ which is 1-exponential w.r.t. P.

$$P(-\log(U_1) > x) = \begin{cases} \exp(-x) & \text{kun } x \geq 0 \\ 1 & \text{kun } x < 0 \end{cases}$$

   $-\log(U_1) \in L^1(P)$ with $E_P(-\log(U_1)) = 1$.

(a) Let $Z_0 = 1$, and

$$Z_t(\omega) = 2^t \prod_{s=1}^{t} U_s(\omega)$$

Show that (Z_t) is a martingale in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$, with $\mathcal{F}_t = \sigma(Z_1, Z_2, \ldots, Z_t) = \sigma(U_1, U_2, \ldots, U_t)$.

(b) Show that $E_P(Z_t) = 1$.

(c) Show that the limit $Z_\infty(\omega) = \lim_{t\to\infty} Z_t(\omega)$ exists P almost surely.

(d) Show that

$$Z_\infty(\omega) = 0 \qquad P\text{-a.s.}$$

Hint Compute first the P-a.s. limit

$$\lim_{t\to\infty} \frac{1}{t} \log(Z_t(\omega))$$

(remember Kolmogorov's strong law of large numbers!).

(e) Show that the martingale $(Z_t(\omega) : t \in \mathbb{N})$ is not uniformly integrable.

(f) Show that $\log(Z_t(\omega))$ is a supermartingale, does it satisfy the assumptions of Doob's martingale convergence theorem ?

(g) At every time $t \in \mathbb{N}$, define the probability measure

$$Q_t(A) := E_P(Z_t \mathbf{1}_A) \qquad \forall A \in \mathcal{F}_t$$

on the probability space $(\Omega, \mathcal{F})$.
Show that the random variables $(U_1, \ldots, U_t)$ are i.i.d. also under Q_t, compute their probability density under Q_t.

7. (Paley's and Littlewood's maximal function) Consider a function in $f(x) \in L^1(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), dx)$.

Define the σ-algebra

$$\mathcal{F}_k = \sigma\big\{ Q_{k,z} = \big(z2^{-k}, (z+1)2^{-k}\big],\ z \in \mathbb{Z}^d \big\} \subseteq \mathcal{B}(\mathbb{R}^d), \quad k \in \mathbb{Z}$$

and the two sided filtration $\mathbb{F} = (\mathcal{F}_k : k \in \mathbb{Z})$ where the dyadic cubes $(Q_{k,z} : z \in \mathbb{Z}^d)$ form a partition of $\mathbb{R}^d$, and the functions

$$f_k(x) = \sum_{z\in\mathbb{Z}^d} \mathbf{1}(x \in Q_{k,z}) \frac{1}{|Q_{k,z}|} \int_{Q_{k,z}} f(y)dy$$

where for $k \in \mathbb{Z}$, $|Q_{k,z}| = 2^{-kd}$ is the Lebesgue measure of the d-dimensional dyadic cube

(a) Show that $f_k(x)$ is an $\mathbb{F}$-martingale w.r.t. Lebesgue measure. Note that the definition of conditional expectation martingales extends directly to the case where we integrate with respect to σ-finite positive measures, where the martingale property in this case means

$$\int_{\mathbb{R}^d} f(x) g_k(x) dx = \int_{\mathbb{R}^d} f(x) g_k(x) dx$$

$\forall k \in \mathbb{Z}$ and $g_k(x)$ bounded and $\mathcal{F}_k$-measurable.

To work with a probability measure, we could take instead with $f(x) \in L^1([0,1]^d, \mathcal{B}([0,1]^d), dx)$.

(b) Show that

$$\lim_{k \to -\infty} f_k(x) = 0, \quad \forall x \in \mathbb{R}^d,$$

but

$$\int_{\mathbb{R}^d} f_k(x)dx = \int_{\mathbb{R}} f(x)dx \quad \text{which can be} \neq 0$$

In particular this means that the Doob's martingale backward convergence theorem does NOT extend to the case of σ-finite measures.

(c) Show that $\lim_{k \to +\infty} f_k(x) = f(x)$ almost everywhere and in L^1.

(d) Define the *maximal function*

$$f^{\square}(x) := \sup_{k \in \mathbb{Z}} f_k(x)$$

Use the martingale maximal inequalities to show that for $1 < p < \infty$

$$\| f^{\square}(x) \|_{L^p(\mathbb{R}^d)} \leq \frac{p}{p-1} \sup_{k \in \mathbb{Z}} \| f_k \|_{L^p(\mathbb{R}^d)} \leq \frac{p}{p-1} \| f \|_{L^p(\mathbb{R}^d)}$$

and

$$cP(|f^{\square}(x)| > c) \leq \sup_{k \in \mathbb{Z}} \| f_k \|_{L^1(\mathbb{R}^d)} \leq \| f \|_{L^1(\mathbb{R}^d)}$$