Stochastic analysis, spring 2014, Exercises-4, 8.10.2014

1. Let $\tau_1(\omega)$ and $\tau_2(\omega)$ stopping times with respect to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in T)$ taking values in T . Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.

   Use the definition of stopping time to show that $\sigma(\omega) = \min(\tau_1(\omega), \tau_2(\omega))$ is a $\mathbb{F}$-stopping time.

2. Let $(M_t(\omega))_{t\in T}$ a martingale with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)$ with $M_0(\omega) = 0$. Here T could be either $\mathbb{R}^+$ or $\mathbb{N}$.

   Define the family of random times $\tau_x : x \in \mathbb{R}$

   $$\tau_x(\omega) = \begin{cases} \inf\{s : M_s \geq x\} & \text{for } x \geq 0 \\ \inf\{s : M_s \leq x\} & \text{for } x < 0 \end{cases}$$

   Show that τ_x is a stopping time.

3. Consider a symmetric random walk in discrete time,

   $$M_n = X_1 + \cdots + X_n$$

   where $(X_k : k \in \mathbb{N})$ are independent and identically distributed Bernoulli random variables with $P(X_n = 1) = P(X_n = -1) = 1/2$.

   (a) Compute $P(M_n = k)$ for $n, k \in \mathbb{N}$.

   (b) For $x \in \mathbb{R}$, use Stirling approximation of the factorial of a large $n \in \mathbb{N}$

   $$n! \sim \exp(-n)n^n\sqrt{2\pi n}$$

   to approximate

   $$P\big(M_{2n} = 2\lfloor x \rfloor\big)$$

   (c) Consider the filtration generated by the random walk $\mathbb{F} = \big(\mathcal{F}_n^X\big)$, with $\mathcal{F}_n^X = \sigma(X_k : 0 \leq k \leq n)$. Show that

   $$M_n, \quad (M_n^2 - n), \text{ and } \exp\big(-\theta M_n\big)\cosh(\theta)^{-n}$$

   are $(P, \mathbb{F})$-martingales, where $\cosh(x) = \big(e^x + e^{-x}\big)/2$.

   (d) Prove the *Markov property*

   $$P\big(M_n = k\big|\mathcal{F}_{n-1}\big)(\omega) = P\big(M_n\big|M_{n-1}\big)(\omega) = P\big(X_n = k - \ell\big)\Big|_{\ell = M_{n-1}(\omega)}$$

   and, for $0 \leq m \leq n$

   $$P\big(M_n = k\big|\mathcal{F}_m\big)(\omega) = P\big(M_n\big|M_m\big)(\omega) = P\big(M_{n-m} = k - \ell\big)\Big|_{\ell = M_m(\omega)}$$

   Hint: use the definition and properties of the conditional expectation.

(e) In discrete time, let τ be a stopping time with respect to the filtration $\mathbb{F}$, then the stopped σ-algebra $\mathcal{F}_\tau$ is defined as

$$\mathcal{F}_\tau = \big\{ A \in \mathcal{F} : \forall t \in \mathbb{N},\ A \cap \{\tau \le t\} \in \mathcal{F}_t \big\}$$

Show that τ itself is $\mathcal{F}_\tau$-measurable. Hint (use the definition of stopping time).

(f) Show the *strong Markov property* of the random walk:

$$\widetilde{M}_n := (M_{\tau+n} - M_\tau)$$

is a symmetric random walk independent from the stopped σ-algebra $\mathcal{F}_\tau$. Hint:

$$A = \bigcup_{k \in \mathbb{N}} A \cap \{\tau = k\}$$

and A is $\mathcal{F}_\tau$-measurable if and only if $\forall k$, $A \cap \{\tau = k\}$ is $\mathcal{F}_k$ measurable. Use the definition of conditional expectation w.r.t. $\mathcal{F}_\tau$.

(g) Consider the stopping time $\sigma(\omega) = \min(\tau_a, \tau_b)$ where $a < 0 < b \in \mathbb{N}$, and the stopped martingales $(M_{t\wedge\sigma})_{t\in\mathbb{N}}$ and $(M^2_{t\wedge\sigma} - t \wedge \sigma)_{t\in\mathbb{N}}$. Show that Doob's martingale convergence theorem applies and

$$\lim_{t\to\infty} M_{t\wedge\sigma}(\omega) = M_\sigma(\omega)$$

exists P-almost surely.

(h) Consider now $(M^2_{t\wedge\sigma} - t \wedge \sigma)$. Use the martingale property together with the reverse Fatou lemma to show that $E(\sigma) < \infty$ which implies $P(\sigma < \infty) = 1$.

(i) For $a < 0 < b \in \mathbb{N}$, compute $P(\tau_a < \tau_b)$.
Hint: a martingale has constant expectation $E_P(M_t) = E_P(M_0)$. This holds also for the stopped martingale $M^\tau_t = M_{t\wedge\tau}$.

4. Let $M_t(\omega) = B_t(\omega)$, $t \in \mathbb{R}^+$, a Brownian motion which is assumed to be $\mathbb{F}$-adapted, and such that for all $0 < s < t$ the increment $(B_t - B_s)$ is P-independent from the σ-algebra $\mathcal{F}_s$.

Note this since by assumption the Brownian motion is $\mathbb{F}$-adapted, it follows that $\mathcal{F}^B_t = \sigma(B_s : 0 \le s \le t) \subseteq \mathcal{F}_t$, which could be strictly bigger.

We have seen in the lectures that

$$B_t,\ M_t = (B^2_t - t) \ \text{ and } Z_t = \exp(aB_t - a^2 t/2)$$

are $(P, \mathbb{F})$-martingales.

(a) Let $\sigma(\omega) = \min(\tau_a(\omega), \tau_b(\omega))$, for $a < 0 < b \in \mathbb{R}$. We will see in the lectures that the Doob martingale convergence theorem applies also to continuous martingales in continuous time. By following the same line of proof as in the random walk case check that $P\big(\sigma < \infty\big) = 1$.

(b) Let $a < 0 < b \in \mathbb{R}$. Compute $P\big(\tau_a < \tau_b\big)$.

Hints: When M is either a Brownian motion or a random walk, the stopped process $M_{t\wedge\sigma}(\omega)$ is a uniformly bounded martingale. To compute $P(\tau_a < \tau_b)$, use first the martingale property

$$E(M_{t\wedge\sigma}) = E(M_0) = 0,$$

then for $t \to \infty$ use the bounded convergence theorem.

(c) Use Doob martingale convergence theorem to show that $Z_\infty = \lim_{t\to\infty} Z_t(\omega)$ exists P almost surely.

(d) Show that $Z_\infty(\omega) = 0$ P-a.s.

Hint: Use the strong law of large numbers to show that

$$\lim_{t\to\infty} \log(Z_t)/t = -1/2, \;\; P \text{ a.s.}$$