Stochastic analysis, fall 2014, Exercises-3, 01.10.14

1. Let $(\dot\eta_n(t) : n \in \mathbb{N})$ an othonormal system in $L^2([0,1], dt)$, with

$$\int_0^1 \dot\eta_n(s)\dot\eta_m(s)ds = \delta_{n,m}$$

for example the Haar system we have used in Paul Lévy's construction of Brownian motion, and let $(G_n(\omega) : n \in \mathbb{N})$ a sequence of i.i.d. Gaussian random variable with $E(G) = 0$ and $E(G^2) = 1$. show that the random functions

$$\sum_{k=1}^n \dot\eta_k(s)G_k(\omega)$$

do not converge in $L^2(\Omega \times [0,1], dP \otimes dt)$. Hint: compute the squared norm

$$E\biggl(\int_0^t \biggl\{\sum_{k=1}^N G_d(\omega)\dot\eta_k(s)\biggr\}^2 ds\biggr)$$

2. Let $\xi(\omega) = (\xi_1(\omega), \dots, \xi_d(\omega)) \in \mathbb{R}^d$ a Gaussian random vector with independetn and identically distributed components standard Gaussian components $\xi_k(\omega) \sim \mathcal{N}(0,1)$, $E_P(\xi_k) = 0$ ja $E_P(\xi_k\xi_\ell) = \delta_{k\ell}$. Let $\mu = (\mu_1, ;\mu_d) \in \mathbb{R}^d$ a deterministic vector ja $A = (A_{ij} : 0 \le i \le j \le d)$ a deterministic $d \times d$ matrix.

   Let $X(\omega) = (\mu + A\xi(\omega)^\top) \in \mathbb{R}^d$.

   (a) Show that $E_P(X) = \mu$ ja $E_P(X_iX_j) - E_P(X_i)E_P(X_j) = \Sigma_{ij}$, jossa $\Sigma = AA^\top$.

   (b) Show that the random vector X has density with respect to the Lebesgue measure in $\mathbb{R}^d$ given by

$$p_X(x) = (2\pi)^{-d/2}\det(\Sigma)^{-1/2}\exp\bigl(-\frac{1}{2}(x-\mu)\Sigma^{-1}(x-\mu)^\top\bigr)$$

   in other words, if $g : \mathbb{R}^d \to \mathbb{R}$ is

$$E_P\biggl(g(X)\biggr) = E_P\bigl(g(\mu + A\xi^\top)\bigr) =$$

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} g\bigl(\mu_1 + \sum_{j=1}^d A_{1j}y_j, \dots, \mu_d + \sum_{j=1}^d A_{1j}y_j\bigr) \prod_{j=1}^d \biggl\{\frac{1}{\sqrt{2\pi}}\exp(-y_j^2/2)\biggr\} dy_1 \dots dy_d =$$

$$\int_{\mathbb{R}^d} g(x_1, \dots, x_d)p_X(x_1, \dots, x_d)dx_1 \dots dx_d$$

   Hint use the change of variables $x = \mu + Ay^\top$. You can assume that $m = n$ and the matrix A is invertible.

3. Let $(X, Y) \sim \mathcal{N}(0, \Sigma)$ jointly Gaussian random vectors with $X(\omega) \in \mathbb{R}^{n_x}$ and $Y(\omega) \in R^{n_y}$, with means $E(X) = \mu_X$ $E(Y) = \mu_Y$, and covariance

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy}^\top & \Sigma_{yy} \end{pmatrix}$$

Use Bayes formula to compute the conditional densities

$$p_{X|Y}(x|Y=y) \quad \text{ja} \quad p_{Y|X}(y|X=x)$$

4. Consider a random variable $Y(\omega) \in L^2(\Omega, \mathcal{F}, P)$. Consider the linear subspace spanned by the random variable $Y(\omega)$.

   $\text{LinearSpan}(Y)\{b + aY(\omega) : a, b \in \mathbb{R}\}$
   $\subset L^2(\Omega, \sigma(Y), P) = \{g(Y(\omega)) : \ g(y) \text{ Borel measurable } \} \cap L^2(\Omega, \mathcal{F}, P)$

   (a) Show that $LinearSpan(Y)$ is a closed subspace of $L^2(\Omega, \mathcal{F}, P)$.

   (b) Let X a random variable in $L^2(\Omega, \mathcal{F}, P)$. Compute the orthogonal projection of X on $LinearSpan(Y)$.
   Hint: you can assume that $E(X) = 0$ and $E(Y) = 0$.

5. Consider a jointly Gaussian pair of random variables (X, Y), with means $E(X) = 0$ and $E(Y) = 0$, and covariance

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{XY}^\top & \Sigma_{YY} \end{pmatrix}$$

   (a) Compute the orthogonal projection of X on $LinearSpan(Y)$ a and show that it coincides with on $E(X|Y)(\omega)$.
   Hint: compute the orthogonal projection by minimizing w.r.t. a, b

$$E\big((b + aY - X)^2\big)$$

   (b) Compute the conditional variance of X given Y, defined as

$$\text{Cov}(X|Y)(\omega) = E(X^2|Y)(\omega) - E(X|Y)(\omega)^2$$

6. Let $0 < s < t < u$, and $(B_r : r \geq 0)$ a standard Brownian motion with $B_0 = 0$.

   Compute the conditional distribution of B_u conditionally on $\sigma(B_s, B_u)$.

7. A d-dimensional Brownian motion is an $\mathbb{R}^d$-valued stochastic process $B_t = (B_t^{(1)}, \dots, B_t^{(d)})$, $t \geq 0$, where the components $B_t^{(k)}$ are independent $\mathbb{R}$-valued standard Brownian motions.

   (a) Let Q be an orthogonal $d \times d$-matrix , which means $QQ^\top = Q^\top Q = Id$, (equivalently $Q^{-1} = Q^\top$). Show that $(QB_t : t \geq 0)$ is a d-dimensional Brownian motion.

   (b) For a matrix $A \in n \times d$, let $X_t = (AB_t) \in \mathbb{R}^n$. Show that $(X_t : t \geq 0)$ is a Gaussian process (all finite dimensionaldistributions are jointly Gaussian), with independent jointly Gaussian increments, i.e. for $0 \leq s \leq t$, $(X_t - X_s) \perp\!\!\!\perp \mathcal{F}_s^X = \sigma(X_r : r \leq s)$.

   (c) compute the covariance $E(X_t^{(i)} X_s^{(j)})$. Compute the stochastic cross variations

$$[X^{(i)}, X^{(j)}]_t = \lim_{n\to\infty} \sum_{t_k^n \in \Pi^n} \big(X^{(i)}_{t_k^n \wedge t} - X^{(i)}_{t_{k-1}^n \wedge t}\big)\big(X^{(j)}_{t_k^n \wedge t} - X^{(j)}_{t_{k-1}^n \wedge t}\big)$$

for any sequence of partitions (Π^n) with $\Delta(\Pi^n, t) \to 0$, where we take limit in probability and the limit does not depend on the particular sequence (Π^n), and we have also P-almost sure convergence when $\sum_{n\in\mathbb{N}} \Delta(\Pi^n, t) < \infty$.