Stochastic analysis, fall 2014, Exercises-2, 17.09.14

1. Consider a standard Brownian motion $(B_t(\omega) : t \geq 0)$ and for $n \in \mathbb{N}$.

   (a) For any sequence of partition $(\Pi^n : n \in \mathbb{N})$ with $\Delta(\Pi_n, t) := \sup_{t_i^n \in \Pi^n} |t_i^n - t_{i-1}^n| \to 0$ For $\alpha \in [0, 1$ define the convex combination $t_i^n(\alpha) := \alpha t_i^n + (1-\alpha) t_{i-1}^n$.

   $$\lim_{n\to\infty} \sum_{i=1}^n \big(B_{t_i^n \wedge t} - B_{t_i^n(\alpha)\wedge t}\big)^2 \to \alpha [B, B]_t = \alpha t$$

   with convergence in probability.

   (b) Use Borel Cantelli lemma to show that we have also P-almost sure convergence when $\sum_n \Delta(\Pi^n, t) < \infty$.

   (c) Under the same assumptions,

   $$\lim_{n\to\infty} \sum_{i=1}^n \big(B_{t_i^n \wedge t} - B_{t_i^n(\alpha)\wedge t}\big)\big(B_{t_i^n(\alpha)\wedge t} - B_{t_{i-1}^n \wedge t}\big) \to 0 \to \alpha [B, B]_t = \alpha t$$

   with convergence in probability, and with P-almost sure convergence when $\sum_n \Delta(\Pi^n, t) < \infty$. Hint: the Brownian motion has independent increments, $(B_t - B_s) \perp\!\!\!\perp (B_u - B_v)$ when $0 \leq s \leq t \leq u \leq v$.

   (d) Show that for $F \in C$

   $$\lim_{n\to\infty} \sum_{i=1}^n F_x(B_{t_i^n(\alpha)})\big(B_{t_i^n \wedge t} - B_{t_{i-1}^n \wedge t}\big) \to \int_0^t F_x(B_s) d\overrightarrow{B}_s + \alpha \int_0^t F_{xx}(B_s) ds$$
   $$= F(B_t) - F(B_s) + \big(\alpha - \frac{1}{2}\big) \int_0^t F_{xx}(B_s) ds$$

   where $\int_0^t F_x(B_s) d\overrightarrow{B}_s$ denotes the forward integral and the convergence is in probability when $\Delta(\Pi_n, t) \to 0$ and P-almost sure when $\sum_n \Delta(\Pi^n, t) < \infty$.

2. Let x_t a continuous path with quadratic variation $[x, x]_t$ among the dyadic sequence of partitions Π_n, and a_t a continuous process with finite variation.

   Use Ito formula to show the integration by parts formula.

   $$x_t a_t = x_0 a_0 + \int_0^t a_t dx_t + \int_0^t x_s da_s$$

   Hint: consider the function $F(x, a) = xa$.

3. Let x_t a continuous path with quadratic variation $[x, x]_t$ among the dyadic sequence of partitions Π_n, and $z_0 > 0$.

   Show that $z_t = z_0 \exp\big(x_t - \frac{1}{2}[x, x]_t\big)$ satisfies the linear pathwise differential equation

$$dz_t = z_t dx_t,$$

which is understood in integral sense

$$z_t = z_0 + \int_0^t z_s \overleftarrow{dx_t}$$

4. What is the quadratic variation of z_t ?
5. Show that $z_t^{-1} = z_0^{-1} \exp\bigl(-x_t + \frac{1}{2}[x,x]_t\bigr)$ satisfies

$$z_t^{-1} = z_0^{-1} - \int_0^t z_s^{-1} dx_s + \int_0^t z_s^{-1} d[x,x]_s$$

   Remarks: note that from the assumptions it follows that z_t is bounded away from zero on any compact interval, which means $1/z_t$ is bounded on compacts.

   Note that by definition $[-x,-x]_t = [x,x]_t$.

6. Let a_t be a continuous path with finite first variation, and z_t as before.

   Show that

$$\xi_t = \left(1 + \int_0^t \frac{1}{z_s} da_s\right) z_t$$

   satisfies the linear inhomogeneous pathwise differential equation

$$d\xi_t = \xi_t dx_t + da_t, \quad \xi_0 = z_0$$

7. Let b_t a continuous path with finite first variation and x_t continuous with quadratic variation $[x,x]_t$ among the dyadic sequence of partitions. Show that

$$\int_0^t a_s dx_s = a_t x_t - a_0 x_0 - \int_0^t x_s da_s = \lim_{\Delta(\Pi)\to 0} \sum_{t_i \in \Pi} a_{t_i} \bigl(x_{t_{i+1}\wedge t} - x_{t_i \wedge t}\bigr)$$

   it is well defined independently of the sequence of partitions.

   Hint: use Abel discrete integration by parts formula for a partition Π and take limit as $\Delta(\Pi) \to 0$.

8. Show that $y_t = \int\limits_0^t a_s dx_s$ has quadratic variation among the dyadic sequence of partitions given by

$$[y,y]_t = \int_0^t a_s^2 d[x,x]_s$$

   Hint:

$$\left(\int_{t_i}^{t_{i+1}} a_s dx_s\right)^2 = \left(a_{t_i}(x_{t_{i+1}} - x_{t_i}) + \int_{t_i}^{t_{i+1}} (x_{t_{i+1}} - x_s) da_s\right)^2$$

   develop the squares and take sum over $t_i \in \Pi \cap [0,t]$ and let $\Delta(\Pi) \to 0$.