Stochastic analysis, Fall 2014, Exercises-1, 10.09.2014

1. Let $(x_t : t \in \mathbb{N})$, $(y_t : t \in N)$. Check Abel discrete integration by parts formula:

$$x_t y_t = x_0 y_0 + \sum_{s=1}^{t} y_{s-1}\Delta x_s + \sum_{s=1}^{t} x_{s-1}\Delta y_s + [x,y]_t$$

   where

$$\Delta x_t = x_t - x_{t-1}, \quad , \Delta y_t = y_t - y_{t-1}, \quad , [x,y]_t = \sum_{s=1}^{t} \Delta x_s \Delta y_s$$

2. Let $x : \mathbb{R}^+ \to \mathbb{R}$ a function. Its total variation (or first-variation) is defined as

$$v_t(x) = \sup_{\Pi} \sum_{t_i \in \Pi} \left| x_{t_i \wedge t} - x_{t_{i-1} \wedge t} \right| \tag{1}$$

   where the supremum is over all finite partitions

$$\Pi = \{0 \le t_0 < t_1 < \cdots < t_n\}$$

   $t \wedge s = \min\{s, t\}$.

   Assume that $s \to x_t$ is right continuous with left limits at all points $s \in [0, t]$. We say that x_s is *cadlag* from the french *Continue a Droite, avec Limite A Gauche.*

   We decompose $x_s = x_s^c + \sum_{r \le s} \Delta x_r$ where the path x_s^c is continuous and there are at most countably many jumps, since $\forall n$ the set

$$\left\{ s \in [0,t] : |\Delta x_s| > 1/n \right\} \tag{2}$$

   is finite, otherwise there would be an accumulation point, and the function x_s would not have left or right limit at that point.

   We can show that

$$v_t(x) = v_t(x^c) + \sum_{r \le t} |\Delta x_r| \tag{3}$$

   Show that

$$v_t(x) = \lim_{\Delta(\Pi) \to 0} \sum_{t_i \in \Pi} \left| x_{t_i \wedge t} - x_{t_{i-1} \wedge t} \right|$$

   where $\Delta(\Pi) = \max\{|t_i - t_{i-1}| : t_i \in \Pi\}$ and Π is a finite partition. This means that the limits exists for any sequence of partitions (Π_n) with $\Delta(\Pi_n) \to 0$ and the limiting value does not depende on the sequence.

   Hint: by triangular inequality, when we refine the partition by adding a point the approximation on the right hand side of (1) does not decrease.

   Show the results first for a continuous path (which will be absolutely continuous on the compact interval) .

Assume $v_T(x) < \infty$, for some $T \in (0, \infty]$. For $t \in [0, T]$, let

$$x_t^{\oplus} = \frac{v_t(x) + x_t - x_0}{2}, \quad x_t^{\ominus} = \frac{v_t(x) - x_t + x_0}{2}$$

Show that $x_t^{\oplus}$ and $x_t^{\ominus}$ are non-decreasing satisfying $x_0^{\oplus} = x_0^{\ominus} = 0$ and

$$x_t = x_0 + x_t^{\oplus} - x_t^{\ominus}, \quad v_t(x) = x_t^{\oplus} + x_t^{\ominus} \tag{4}$$

Show that if

$$x_t = x_0 + y_t^{\oplus} - y_t^{\ominus} \tag{5}$$

with $y_t^{\oplus}$ and $y_t^{\ominus}$ non-decreasing satisfying $y_0^{\oplus} = y_0^{\ominus} = 0$, then

$$v_t \le y_t^{\oplus} + y_t^{\ominus}$$

Show that the decomposition (4) is minimal among the decompositions (5), meaning that

$$a_t := y_t^{\oplus} - x_t^{\oplus} = y_t^{\ominus} - x_t^{\ominus}$$

is non-decreasing.

3. Assume that x_t is continuous a path with quadratic variation $[x]_t$ among the sequence of partitions $(\Pi_n)_{n \in \mathbb{N}}$, and let a_t a continuous function with finite first variation on compact intervals, that is $v_t(a) < \infty$. Let $F(x, a)$ a $C^{2,1}$ function, with continuous partial derivatives $F_a(x, a), F_x(x, a)$ and $F_{xx}(x, a)$.

   Use Taylor expansion, uniform continuity, and the vague convergence definition of the quadratic variation to show in details the extended Ito-Föllmer formula

$$\begin{aligned} F(x_t, a_t) - F(x_0, a_0) - \int_0^t F_a(x_s, a_s) da_s - \frac{1}{2} \int_0^t F_{xx}(x_s, a_s) d[x]_s = \\ = \int_0^t F_x(x_s, a_s) d\overleftarrow{x}_s = \lim_{n \to \infty} \sum_{t_i^n \in \Pi_n} F_x(x_{t_i^n}) \big(x_{t_{i+1}^n \wedge t} - x_{t_i^n \wedge t} \big) \end{aligned} \tag{6}$$

   where the last equality defines the pathwise integral.

   Hint: Write the telescopic sum with and use Taylor

$$\begin{aligned} & F(x_{t_{i+1}}, a_{t_{i+1}}) - F(x_{t_i}, a_{t_i}) = \\ & F(x_{t_{i+1}}, a_{t_{i+1}}) - F(x_{t_{i+1}}, a_{t_i}) + F(x_{t_{i+1}}, a_{t_i}) - F(x_{t_i}, a_{t_i}) = \\ & F_a(x_{t_{i+1}}, a_{\tau_i})(a_{t_{i+1}} - a_{t_i}) + F_x(x_{t_i}, a_{t_i})(x_{t_{i+1}} - x_{t_i}) + \frac{1}{2} F_{xx}(x_{t_i}, a_{t_i})(x_{t_{i+1}} - x_{t_i})^2 \\ & + \frac{1}{2} \big(F_{xx}(x_{\sigma_i}, a_{t_i}) - F_{xx}(x_{t_i}, a_{t_i}) \big)(x_{t_{i+1}} - x_{t_i})^2 \end{aligned}$$

   for some $t_i \le \tau_i \le t_{i+1}, \quad t_i \le \sigma_i \le t_{i+1}$ which exist by the mid-point theorem of elementary analysis.

4. Assume that x_t is a continuous path with $x_0 = 0$, and quadratic variation $[x]_t = t$, among the dyadic sequence of partitions $\mathcal{D} = \big(t^n_k = k2^{-n} : k \in \mathbb{N}\big)_{n\in\mathbb{N}}$, and let $a_t = \exp(t)$. Use the change of variable formula of classical Riemann-Stieltjes integrals and Ito-Föllmer formula (6) to compute the integral representation of

   - $\sin(a_t)$,
   - $\sin(x_t)$,
   - $\sin(a_t x_t)$.

5. What is the first variation of $\sin(a_t)$?

   What is the quadratic variation of $\sin(x_t)$?

   What is the quadratic variation of $\sin(a_t x_t)$?

   Show that $\sin(x_t)$ and $\sin(a_t x_t)$ have infinite first variation.

6. Let $(x_t : t \in \mathbb{R}_+)$ continuous with quadratic variation $[x]_t = [x,x]_t$, ja $t \mapsto a_t \in C^1(\mathbb{R}_+,\mathbb{R})$.

   (a) Show that a_t has quadratic variation $[a,a]_t = 0$, and the cross variation $[x,a]_t = 0$.

   (b) Let $y_t = a_t + x_t$, and recall that $[y,y] = [a,a]+2[x,a]+[x,x] = [x,x]$. Use Ito formula to compute the pathwise Ito Föllmer integrals

   i.
   $$\int_0^t x_s^n dx_s, \quad n \in \mathbb{N}$$

   ii.
   $$\int_0^t \exp(\sigma x_s^n) dx_s, \quad n \in \mathbb{N}$$

   iii.
   $$\int_0^t \sin(\sigma x_s) dx_s,$$

   iv.
   $$\int_0^t \cos(\sigma x_s) dx_s,$$

   v.
   $$\int_0^t y_s^n dy_s$$

   vi.
   $$\int_0^t \exp(\sigma y_s^n) dy_s$$

vii.

$$\int_0^t a_s^n da_s$$

viii.

$$\int_0^t \exp(\sigma a_s^n) da_s$$

ix.

$$\int_0^t \sin(\sigma a_s) da_s,$$

x.

$$\int_0^t \cos(\sigma a_s) da_s,$$

xi.

$$\int_0^t y_s^n dx_s$$

xii.

$$\int_0^t \exp(\sigma y_s^n) dx_s$$

(c) Compute also the quadratic variation of those integrals. Recall that the pathwise integral $t \mapsto \int_0^t y_s dx_s$ when it exists has quadratic variation $\int_0^t y_s^2 d[x,x]_s$.