Stochastic analysis, Fall 2014, Exercise 9,3.11.2013

We apply stochastic calculus to the theory of random matrices.

Consider first a $d \times d$ Hermitian matrix $A = A^*$ which means $A_{ij} = \bar{A}_{ji}$ where forÂăÂă$\bar{z}$ is the conjugate of $z \in \mathbb{C}$, and $A_{ii} \in \mathbb{R}^+$.

The Hermitian matrix has orthonormal eigenvalues

$$Au_i = \lambda_i u_i$$

with $\lambda_i \in \mathbb{R}^+$ and $u_i^* u_j = \delta_{ij}$ where $u_i^* u_j = \sum_{k=1}^d \bar{u}_{ik} u_{jk}$

Assume now that $A = A(t)$ depending on a time parameter. Then by differentiating w.r.t. t

$$A(t)u_i(t) = \lambda_i(t)u_i(t)$$

we obtain

$$dA(t)u_i(t) + A(t)du_i(t) = d\lambda_i(t)u_i(t) + \lambda_i(t)du_i(t) \tag{1}$$

and since

$$du_i^*(t)u_i(t) + u_i^*(t)du_i(t) = 0 \Longrightarrow u_i^*(t)du_i(t) = 0$$

which means $du_i(t)$ is orthogonal to $u_i(t)$, which implies by taking inner product of (1) with u_i

$$d\lambda_i(t) = u_i^* dA(t)u_i \quad \text{Hadamard first variation formula}$$

By differentiating twice in the same way we get the formula

$$d^2\lambda_k = u_k^* d^2 A u_k + 2\sum_{j \neq k} \frac{|u_j^* dA u_k|^2}{\lambda_k - \lambda_j}$$

We now consider Dyson Brownian motion.

The Gaussian Unitary Ensemble (GUE) of random matrices is the set of Hermitian $d \times d$ matrices with

$$G_{ij} = \bar{G}_{ij} = X_{ij} + \sqrt{-1}Y_{ij} \quad i \neq j,$$

and X_{ij}, Y_{ij} are i.i.d $\mathcal{N}(0, 1/2)$ and $G_{ii} \sim \mathcal{N}(0,1)$. so that

$$p(G) = c(d)\exp\bigl(-\frac{d}{2}\text{Trace}(G^2)\bigr)$$

We consider now a matrix valued Complex Gaussian process, where

$$G_{ij}(t) = \bar{G}_{ij}(t) = X_{ij}(t) + \sqrt{-1}Y_{ij}(t) \quad i \neq j,$$

and $\sqrt{2}X_{ij}(t), \sqrt{2}Y_{ij}(t)$ and G_{ii} are independent standard Brownian motions.

Now the ordered eigenvalues of $G(t)$Âă

$0 \le \lambda_1(t) < \lambda_2(t) < \ldots \lambda_d(t)$

form a d-dimensional process $\lambda(t)$ called Dyson Brownian motiom.

Note that

$$d\lambda(t) =$$