HW. 3
EX 1

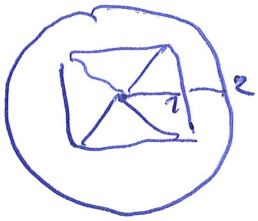
1) $U = (-1,1)\times(-1,1)$

we can see that u is Lipschitz

$$|u(x)-u(y)| \le |x-y| \quad , \forall x,y \in U$$

Let us define extension

$$\tilde{u} = \begin{cases} u, & x \in U \\ 0, & x \in B(0,2)\setminus U \end{cases}$$



we see that $\tilde{u} : B(0,2) \to R$

is Lipschitz with smooth boundary

Using Theorem 4 (Characterization of $W^{1,\infty}$)

$\tilde{u} : B(0,2) \to R$ is Lipschitz $\iff \tilde{u} \in W^{1,\infty}(B(0,2))$

$\Rightarrow B(0,2)$ is bounded $\Rightarrow \tilde{u} \in W^{1,p}(B(0,2))$, $1 \le p \le \infty$

$u = \tilde{u}|_U \Rightarrow u \in W^{1,p}(U)$, $1 \le p \le \infty$

2) $\quad U \in H^2(U) \cap H^1_0(U)$

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then we have

$U_n \xrightarrow{H^1} U$, and $U_n \in C_0^\infty(U)$

and

$W_k \xrightarrow{H^2} U$, and $W_k \in C^\infty(U)$

then we know that

$$\partial_{x_j} U_n \xrightarrow{w} \partial_{x_j} U$$

So Let $k \in \mathbb{N}$

$$\lim_{n\to\infty} \left\langle \partial_{x_j} U_n , \partial_{x_j} W_k \right\rangle = \left\langle \partial_{x_j} U , \partial_{x_j} W_k \right\rangle$$

also we know that

$$\partial_{x_j} W_k \xrightarrow{w} \partial_{x_j} U$$

so $\displaystyle\lim_{k\to\infty} \left\langle \partial_{x_j} U , \partial_{x_j} W_k \right\rangle = \left\langle \partial_{x_j} U , \partial_{x_j} U \right\rangle$

On the other Hand
Integ By Parts

$$\left\langle \partial_{x_j} U_n , \partial_{x_j} W_k \right\rangle = -\left\langle U_n , \partial_{x_j}\partial_{x_j} W_k \right\rangle \leq \|U_n\|_{L^2(U)} \|\partial_{x_j}\partial_{x_j} W_k\|$$

this Hold's for any $n \in \mathbb{N}$ and Limit's Exist's

So:

$$\left\langle \partial_{x_j} U , \partial_{x_j} W_k \right\rangle \leq \|U\|_{L^2(U)} \|\partial_{x_j}\partial_{x_j} W_k\|$$

also thie inequality Hold's for Any $k \in \mathbb{N}$ and Limit's Exist's

So:

$$\left\langle \partial_{x_j} U , \partial_{x_j} U \right\rangle \leq \|U\|_{L^2} \|\partial_{x_j}\partial_{x_j} \cancel{W} \|_{L^2}$$

$$\|Du\|^2_{L^2(U)} = \int_U |Du|^2 \, dx = \sum_{j=1}^{n} \langle \partial_{x_j} u, \partial_{x_j} u \rangle \leq \sum_{j=1}^{n} \|u\|_{L^2} \|\partial_{x_j} \partial_{x_j} u\|_{L^2}$$

$$\leq n \|u\|_{L^2} \|D^2 u\|_{L^2}$$

So

$$\|Du\|_{L^2(U)} \leq \sqrt{n} \, \|u\|^{1/2}_{L^2} \|D^2 u\|^{1/2}_{L^2}$$

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EX 2

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4) U is connected and $u \in W^{1,p}(U)$

and $Du = 0$ a.e $x \in U$

Let's choose set's

$U_1 \subset\subset U_2 \subset\subset U_3 \subset\subset \cdots \subset\subset U$

that $U = \bigcup_{j=1}^{\infty} U_j$

Let's fix $j \in \mathbb{N}$ and

choose $0 < \varepsilon < \operatorname{Dist}(U_j, \mathbb{R}^n \setminus U)$

then $\forall x \in U_j$

$$D(\eta_\varepsilon * u)(x) = \eta_\varepsilon * Du(x) = 0$$

Because $\eta_\varepsilon * u \in C^\infty(\bar{U}_j)$

$$\eta_\varepsilon * u = C^\varepsilon$$

Let $\varepsilon \to 0$, AND we see that

~~a.e x~~

$$u(x) = \lim_{\varepsilon \to 0} \eta_\varepsilon * u = C_j \text{ , a.e } x \in \bar{U}_j$$

Because $\bar{U}_1 \subset\subset \bar{U}_2 \subset\subset \cdots \subset\subset U_j \subset\subset \cdots$

we see that $u(x) = C_1$ a.e $x \in \bigcup_{j=1}^{\infty} \bar{U}_j = U$

5) we say that $u \in H_0^1(U)$ is a weak solution of the Boundary-value problem

$$\begin{cases} \Delta u = f & x \in U \\ u = 0 & , x \in \partial U \end{cases}$$

where U is open, bounded subset of $\mathbb{R}^N$

and $u : \bar{U} \to \mathbb{R}$ is unknown solution,

if $B[u,v] = (f,v)_{L^2(U)}$

For all $v \in H_0^1(U)$ and $f \in L^2(U)$.

where Bilinear form is:

$$B[u,v] = -\int_U \sum_{j=1}^{n} \partial_j u \, \partial_j v \, dx = \int_U f v \, dx$$

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6) Look the Remark from Evans!! (End of 6.2.1)

... If the Bilinear form is symmetric

..... Riesz Representation Theorem

Directly applies... ~ ~

But LAX-MILGRAM Theorem is Primarily significant in that it Does not require symmetry

Lemma let $f \in C_0^\infty(U) \cap W^{1,p}(U)$
$g \in C^\infty(\bar{U}) \cap W^{2,p}(U)$

$$\int_U \partial_j f \, \partial_j g \, |Dg|^{p-2} \, dx = - \int_U f \, \partial_j \left(\partial_j g \left(\sum_{k=1}^n (\partial_k g)^2 \right)^{\frac{p-2}{2}} \right) dx$$

$$= - \int_U f \left(\partial_j^2 g \, |Dg|^{p-2} + \partial_j g \left(\frac{p-2}{2} \right) |Dg|^{p-4} \left(\sum_{k=1}^n 2 \partial_k g \, \partial_{jk} g \right) \right) dx$$

By Def. we know that $|\partial_j^2 g| \leq |D^2 g|$, $|\partial_j g| \leq |Dg|$
$|\partial_{jk} g| \leq |D^2 g|$

$$\Rightarrow \underbrace{\int_U \partial_j f \, \partial_j g \, |Dg|^{p-2} \, dx}_{\text{“}\ominus} \leq \left(2n \left(\frac{p-2}{2} \right) + 1 \right) \int_U |f| \, |Dg|^{p-2} \, |D^2 g| \, dx$$

using Holder we get:

$$\ominus \leq C(n,p) \left(\int_U |Dg|^p \right)^{\frac{p-2}{p}} \left(\int |f|^{p/2} |D^2 g|^{p/2} \right)^{2/p}$$

using Holder again:

$$\ominus \leq C(n,p) \left(\int_U |Dg|^p \right)^{\frac{p-2}{p}} \left(\left(\int_U |f|^p \right)^{1/2} \left(\int_U |D^2 g|^p \right)^{1/2} \right)^{2/p}$$

We HAVE an estimate

$$\int_U \partial_j f \partial_j g |Dg|^{p-2} dx \leq C(n,p) \|Dg\|^{p-2}_{L^p} \|f\|_{L^p} \|D^2 g\|_{L^p} < \infty$$

$u \in W^{2,p}(U) \cap W_0^{1,p}(U)$ the we have

$u_n \xrightarrow{W^{1,p}} u$, $u_n \in C_0^\infty(U) \cap W^{1,p}$

$w_k \xrightarrow{W^{2,p}} u$, $w_k \in C^\infty(\bar{U}) \cap W^{2,p}$

Let us Define

$$L(u_n, w_k) := \int_U \partial_j u_n \partial_j w_k |Dw_k|^{p-2} dx$$

we see that

$$L(\cdot, w_k) : C_0^\infty(U) \to \mathbb{R}$$

is LINEAR and BounDed using the estimate from lemma

So we CAN EXTEND

$L(\cdot, w_k) : H_0^1(U) \to R$ and we ~~see~~ HAVE an new estimate ;

$$L(u, w_k) \leq C(n,p) \|u\|_{L^2} \|Dw_k\|^{p-2}_{L^p} \|D^2 w_k\|_{L^p}$$

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Because $w_k \xrightarrow{W^{2,p}(U)} u$

w_k is a Cauchy in $W^{2,p}(U)$

and

$$\left| L(u,w_k) - L(u,w_l) \right| \le \left| \int_U \left(\partial_j u\, \partial_j w_k |Dw_k|^{p-2} - \partial_j u\, \partial_j w_l |Dw_l|^{p-2}\right) dx \right|$$

$$\le \underbrace{\int_U |\partial_j u| |\partial_j w_k - \partial_j w_l| |Dw_k|^{p-2}\, dx}_{= I}$$

$$+ \underbrace{\int_U |\partial_j u| |\partial_j w_l| \left| |Dw_k|^{p-2} - |Dw_l|^{p-2} \right| dx}_{II}$$

using Holder

$$I \le \left(\int_U |\partial_j u|^{p/2} |\partial_j w_k - \partial_j w_l|^{p/2} \right)^{2/p} \left(\int_U |Dw_k|^p \right)^{p/p-2} \Rightarrow$$

using Holder

$$I \le \left(\int_U |\partial_j u|^p \right)^{\frac{1}{p}} \left(\int_U |\partial_j w_k - \partial_j w_l|^p \right)^{1/p} \left(\int_U |Dw_k|^p \right)^{p/p-2}$$

$$I \le \|\partial_j u\|_{L^p} \|\partial_j w_k - \partial_j w_l\|_{L^p} \|Dw_k\|_{L^p}^{p/p-2}$$

Note $p > 2$

We Have an Estimate

$$|Dw_k(x)|^{p-2} - |Dw_n(x)|^{p-2} \le C|Dw_k(x) - Dw_n(x)|^{p-2}$$

a.e $x \in U$

then we have for $\overline{II}$

$$\overline{II} \le \int_U |\partial_j u| \partial_j w_n| |Dw_k - Dw_L|^{p-2} dx$$

using the same Hölder as for $\overline{I}$

$$\overline{II} \le \|\partial_j u\|_{L^p} \|\partial_j w_k\|_{L^p} \|Dw_k - Dw_L\|_{L^p}^{p/p-2}$$

Because ~~[illegible]~~

$\partial_j w_k$ — is Cauchy in $L^p(U)$

$\Rightarrow \|\partial_j w_k\|_{L^2} \le C$

Dw_k — is Cauchy in $L^p(U)$

$\|Dw_k\|_{L^2}^{p/p-2} \le C$

we see that

$\{L(u, w_k)\}$ is Cauchy in $\mathbb{R}$

We DID HAVE AN ESTIMATE

$$L(u, w_k) \leq C(n,p) \|u\|_{L^2} \|Dw_k\|^{p-2} \|D^2 w_k\| \quad , \quad \forall k$$

$$\lim_{k\to\infty} L(u, w_k) = \int_U \partial_j u \, \partial_j u \, |Du|^{p-2} \, dx \leq C(n,p) \|u\|_{L^2} \|Du\|^{p-2} \|D^2 u\|$$

$\blacksquare$