$U \subset \mathbb{R}^n$ BOUNDED C^2-DOMAIN, $\begin{cases} \Delta^2 u = f, & x \in U \\ u = \frac{\partial u}{\partial \nu} = 0, & x \in \partial U \end{cases}$

Let $u, \varphi \in C_0^\infty(U)$ then we see

$$\int_U \Delta^2 u \varphi \, dx = \int_U \Delta(\Delta u) \varphi \, dx = -\int_U \nabla(\Delta u) \cdot \nabla \varphi \, dx = \int_U \Delta u \Delta \varphi \, dx$$

we Look solutions in following form:

$$\begin{cases} B : H_0^2(U) \times H_0^2(U) \to \mathbb{R} \\ B[u,v] = \int_U \Delta u \Delta v \, dx = \int_U f v \, dx \end{cases}$$

Part 1: $|B[u,v]| \leq \int_U |\Delta u||\Delta v| \, dx \leq \|\Delta u\|_{L^2(U)} \|\Delta v\|_{L^2(U)}$

$$\leq \|u\|_{H_0^2(U)} \|v\|_{H_0^2(U)}$$

Part 2

$$\|u\|^2_{H_0^2(U)} = \|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)} + \|D^2 u\|^2_{L^2(U)}$$

using Poincare we get:

$$\|u\|^2_{H_0^2(U)} \leq (C+1)\|Du\|^2_{L^2(U)} + \|D^2 u\|^2_{L^2(U)}$$

Integrate by Part's to get :

$$\int_U |Du|^2 dx = \int_U Du \cdot Du \, dx = -\int_\Omega u \Delta u \, dx \overset{C.S}{\leq} \|u\|_{L^2} \|\Delta u\|_{L^2}$$

(Cauchy with ε)

$$\leq \varepsilon \|u\|_{L^2}^2 + \frac{\|\Delta u\|_{L^2}^2}{4\varepsilon} \leq \varepsilon C \|Du\|_{L^2}^2 + \frac{1}{4\varepsilon} \|\Delta u\|_{L^2}^2$$

Let $\varepsilon = \frac{1}{2C}$

$$\Rightarrow \|Du\|_{L^2}^2 \leq C \|\Delta u\|_{L^2}^2$$

then is left to deal $\|D^2 u\|_{L^2(U)}^2$

Integ By Part's and assume that $u \in C_0^\infty(U)$

$$\int_U |D^2 u|^2 dx = \int_U \sum_{i,j}^{n} (\partial_{x_j} \partial_{x_i} u)^2 dx = \sum_{i,j} \int_U \partial_{x_j} \partial_{x_i} u \, \partial_{x_j} \partial_{x_i} u \, dx$$

$$= -\sum_{i,j} \int_U \partial_{x_i} u \, \partial_{x_j} \partial_{x_j} \partial_{x_i} u \, dx = \sum_{i,j} \int_U \partial_{x_i} \partial_{x_i} u \, \partial_{x_j} \partial_{x_j} u \, dx$$

$$= \int_U \left(\sum_{i=1}^{n} \partial_{x_i} \partial_{x_i} u \right) \left(\sum_j^n \partial_{x_j} \partial_{x_j} u \right) dx = \int_U |\Delta u|^2 dx$$

Putting these together we Have Coersivity

$$\|u\|_{H_0^2(U)} \leq C \left((C+1) \|\Delta u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \right) = \tilde{C} \|\Delta u\|_{L^2}^2$$

So $\frac{1}{\tilde{C}} \|u\|_{H_0^2(U)}^2 \leq \|\Delta u\|_{L^2}^2 = B[u,u]$

So LAX-Milgram SAY'S ☺

U is Bounded

we have a weak Form $, u \in H^1(U) , f \in L^2(U)$

1) $$\int_U Du \cdot Dv \, dx = \int_U f v \, dx \quad , \forall v \in H^1(U)$$

Prove that 1 has a weak solution $u \in H^1(U)$

if and only if

2) $$\int_U f \, dx = 0$$

$1^\circ \Rightarrow$ U is a weak solution
Because U is Bounded $1 \in H^1(U)$

So $$\int_U Du \cdot D1 \, dx = 0 = \int_U f \cdot 1 \, dx \quad \square$$

$2^\circ \Leftarrow$ ~~We have two ...~~

first we assumed that

a) $\int_U f \, dx = 0$. so ~~We ...~~

we Have a Linear functional T,

$$T : L^2(U) \to \mathbb{R} \quad , \quad T(f) := \int_U 1 \cdot f \, dx$$

HW6. EX 2. Page 2

a) gives us $T(f) = 0$

so we see that $T^{-1}(\{0\}) = \left\{ f \in L^2(U) : \int_U f\,dx = 0 \right\}$

is a closed supspace. Let's Denote it

$T^{-1}(\{0\}) = H_{f_0}$ (and also is a Hilbert)

If u is a weak solution, then we see from the weak form that also

$\tilde{u} := u + c$ is a weak solution.

So because U is bounded and u is assumed to be in $H^1(U)$ $\int_U u\,dx < \infty \Rightarrow$ so we pick c so that

b) $\int_U \tilde{u}\,dx = 0$ using the same

"Trick" we define functional

$$T : H^1(U) \to \mathbb{R}, \quad T(\tilde{u}) := \int_U 1 \cdot \tilde{u}\,dx = 0$$

and define a subspace that is closed (so Hilbert)

$$T^{-1}(\{0\}) = H_{\tilde{u}} = \left\{ \tilde{u} \in H^1_0(U) : \int_U \tilde{u}\,dx = 0 \right\}$$

HW 6. EX 2 page 3

then we Define our Problem in weak sense

$$B[u,v]: H_{\tilde{U}} \times H_{\tilde{U}} \to \mathbb{R} \quad , f \in H_{S_0}$$

$$B[u,v] = \int_U Du \cdot Dv\, dx = \int_U f v\, dx$$

First part

$$|B[u,v]| \leq \int_U |Du \cdot Dv|\, dx \leq \|Du\|_{L^2(U)} \|Du\|_{L^2(U)} \leq \|u\|_{H\tilde{U}} \|v\|_{H\tilde{U}}$$

Second Part. Here we us Poincare's inequality in following form:

let U Be Bounded and connected C^1-DOMAIN and U open in $\mathbb{R}^N$ then

$$\|u - (u)_U\|^2_{L^2(U)} \leq C \|Du\|^2_{L^2(U)}$$ for each

function $u \in H^1(U)$, where $(u)_U = ⨍_U u\, dy$

So using that and $\int_U \tilde{u}\, dx = 0$, we get:

$$\|\tilde{u}\|^2_{L^2(U)} = \int_U \tilde{u}^2 dx = \int_U \left(\tilde{u} - \frac{1}{|U|}\int_U \tilde{u}(y)\, dy\right)^2 dx \leq C \int_U |Du|^2 dx$$

$$\leq C B[u,u]$$ so

$$\|\tilde{u}\|^2_{H^1(U)} = \|\tilde{u}\|^2_{L^2} + \|Du\|^2_{L^2} \leq C B[u,u] + B[u,u]$$

$$\Rightarrow \frac{1}{C+1} \|\tilde{u}\|_{H^1(U)} \leq B[u,u]$$

So Lax-Milgram and we are done

HW 6.3 Page 7

~~6.6 #5~~

Let $u \in H^1(\mathbb{R}^n)$ have a compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n, \tag{1}$$

where $f \in L^2(\mathbb{R}^n)$, and $c : \mathbb{R} \to \mathbb{R}$ is smooth, with $c(0) = 0$ and $c' \geq 0$. Prove $u \in H^2(\mathbb{R}^n)$.

Hint. Mimic the proof of interior regularity Theorem, but without the cutoff function $\zeta(\cdot)$.

Solution. We use notation of Sect. 6.3.1, Th.1. Also, let $R > 0$ be such that

$$\operatorname{supp}(u) \subset B_R(0). \tag{2}$$

Define weak solution as following: $u \in H^1(\mathbb{R}^n)$ is a weak solution of (1) if

$$\int_{\mathbb{R}^n} Du \cdot Dv + c(u)v \, dx = \int_{\mathbb{R}^n} fv \, dx \tag{3}$$

for all $v \in H^1(\mathbb{R}^n)$.

Remark. *Strictly speaking, for the definition of weak solution given above, we need an extra assumption on $c(\cdot)$ to assure that that $c(u)v \in L^1_{loc}(\mathbb{R}^n)$ (the last inclusion would be sufficient for (3) since u has compact support and $c(0) = 0$, i.e. integration in the term $c(u)v$ can be restricted to $B := B_R(0)$, where R is from (2)). Example $c(t) = e^{t^2} - 1$ shows that some assumption might be needed.*

From Sobolev inequalities $u, v \in L^{\frac{2n}{n-2}}(B)$ if $n > 2$, and $u, v \in L^p(B)$ for any $p \in [1, \infty)$ if $n = 2$. Thus we need to have $c(u) \in L^{\frac{2n}{n+2}}(B)$ if $n > 2$, and $c(u) \in L^p(B)$ for some $p \in (1, \infty)$ if $n = 2$. Then it is sufficient to assume that $|c(t)| \leq Ct^{\frac{n+2}{n-2}}$ if $n > 2$, and $|c(t)| \leq Ct^M$ for some $M \geq 0$ if $n = 2$.

Note also, that, at least for $n > 2$, the above assumptions do not imply that $c(u) \in L^2(\mathbb{R}^n)$, i.e. the assertion in the problem does not follow from the regularity results for linear equations by considering $-\Delta u = g$ where $g = f - c(u)$.

We choose $h \neq 0$ with $|h| \leq 1$, $k \in \{1, \dots, n\}$

$$v = -D_k^{-h} D_k^h u.$$

Then $v \in H^1(\mathbb{R}^n)$ and has compact support. Using this v in (3), and using integration by parts and the "difference quotients integration by parts" formula, and using that u has compact support, get

$$\int_{\mathbb{R}^n} \left(|D_k^h Du|^2 + D_k^h(c(u)) D_k^h u \right) dx = -\int_{\mathbb{R}^n} f\left(D_k^{-h} D_k^h u \right) dx. \tag{4}$$

We calculate:

$$(D_k^h(c(u)))(x) = \frac{c(u(x + he_k)) - c(u(x))}{h} = \eta_h(x) \frac{u(x + he_k) - u(x)}{h} = \eta_h(x) D_k^h u(x),$$

HW 6 Ex 3 Page 2

where

$$\eta_h(x) = \int_0^1 c'(ta + (1-t)b)\,dt, \quad \text{where } \; a = u(x+he_k),\; b = u(x).$$

Thus $c' \geq 0$ implies $\eta_h(x) \geq 0$, and we get

$$D_k^h(c(u))D_k^h u = \eta_h(x)(D_k^h u(x))^2 \geq 0. \tag{5}$$

Also, using that $0 < |h| \leq 1$ and applying Theorem 3 from Sect. 5.8.2 in the region $B_{R+1}(0)$

$$\left|\int_{\mathbb{R}^n} f\left(D_k^{-h} D_k^h u\right) dx\right| = \left|\int_{B_{R+1}(0)} f\left(D_k^{-h} D_k^h u\right) dx\right| \leq \varepsilon \int_{\mathbb{R}^n} |D\, D_k^h u|^2 dx + \frac{C}{\varepsilon}\int_{\mathbb{R}^n} f^2 dx. \tag{6}$$

From (4), (5), (6)

$$\int_{\mathbb{R}^n} |D_k^h Du|^2 \leq \varepsilon \int_{\mathbb{R}^n} |D_k^h Du|^2 dx + \frac{C}{\varepsilon}\int_{\mathbb{R}^n} f^2 dx.$$

Choosing $\varepsilon = \frac{1}{2}$, get

$$\int_{\mathbb{R}^n} |D_k^h Du|^2 \leq C \int_{\mathbb{R}^n} f^2 dx.$$

This is true for each $k \in \{1, \ldots, n\}$, $0 < |h| < 1$. Thus, applying Theorem 3 from Sect. 5.8.2 in the region $B_{R+1}(0)$, get

$$\int_{B_{R+1}(0)} |D^2 u|^2 \leq C \int_{\mathbb{R}^n} f^2 dx,$$

thus

$$\int_{\mathbb{R}^n} |D^2 u|^2 = \int_{B_{R+1}(0)} |D^2 u|^2 \leq C \int_{\mathbb{R}^n} f^2 dx.$$

Combining this with $u \in H^1(\mathbb{R}^n)$, get $u \in H^2(\mathbb{R}^n)$.

U is Bounded. C^1-Domain and connected

Let's assume that $u, v \in C^2(\bar{U})$.

Problem $\begin{cases} -\Delta u = 0, & x \in U \\ \partial_\nu u = 0, & x \in \partial U \end{cases}$

∴ $-\int_U \Delta u v \, dx = 0$ then using Green's (ii)

we get

$$\int_U Du \cdot Dv \, dx = -\int_U \Delta u v \, dx + \int_{\partial U} \frac{\partial u}{\partial \nu} v \, dS = -\int_U \Delta u v \, dx$$

So we have weak formulation for problem as in the 2. Problem

$B[u,v] = \int_U Du \cdot Dv \, dx = 0$, where we

defined, that, $u, v \in H_{\tilde{U}} \subset H^1(U)$

Lax-Milgram gives a unique solution

~~M~~ $u \in H_{\tilde{U}}$ for which $\int_U u \, dx = 0$

But because when $u \equiv 0$

$B[0,v] = 0 \Rightarrow u \equiv 0$ is that

solution in a weak sence.

using EX. 2. we see that

all constant's are as well solutions

of the problem.

Also when we use EVANS Theorem 3

in chapter 6.3, we see that

when $u \in H^1$ is a weak solution of the

elliptic PDE, and $a^{ij}, b^i, c \in C^\infty(U)$ and $f \in C^\infty(U)$

then $u \in C^\infty(U)$ so all smooth

solutions are constants

---

6) suppose $u \in C^2(U) \cap C(\bar{U})$

Properties of harmonic functions

say's that if u is ~~connected~~ harmonic within U
then:

i) $\max_{\bar{U}} u = \max_{\partial U} u$ and

ii) if U is connected and there

exist a point $x_0 \in U$ such that

$u(x_0) = \max_{\bar{U}} u$ then

u is constant within U

Let's assume that $\exists\, \tilde{x} \in \partial U$ s.t

$$u(\tilde{x}) > u(x) \quad , \forall x \in U$$

So Hopf's lemma tell's us

(with ~~the~~ assumption that we have C^2-Boundary)
(interior ball condition at $\tilde{x}$)

then $\frac{\partial u}{\partial \nu}(\tilde{x}) > 0$ $\lightning$

we asumed that $\partial_\nu u = 0$, on ∂U

$\Rightarrow$ So there is $x_0 \in U$ for

$u(x_0) = \max\limits_{\partial U} u$ $\Rightarrow$ and u is constant

HW 6 EX 5

Chapter 6.5 & Theorem 2 from EVANS say's

DIM = 1

HW6 . EX 6

Using Weyl's Formula we can calculete:

$$V = (2\pi)^d \lim_{R \to \infty} \frac{N(R)}{R^{d/2}}$$

, where $N(R)$ is number of eigenvalues smaller than R.

Let λ_n be some eigenvalues

then $\Delta u = -\lambda_n u$

Gauss-green gives:

$$\int_U \Delta u \, dx = \int_{\partial U} \nabla u \cdot \hat{n} \, dS \Rightarrow$$

$$\int_U u \, dx = -\frac{1}{\lambda_n} \int_{\partial U} \nabla u \cdot \hat{n} \, dS$$

So if we measure the neuman-data respect to λ_n "eigenstate" we can calculet $\int_U u\,dx$