HW 5 - EX 1

$u \in H^s(\mathbb{R}^n)$ and $s > n/2$

$$\Rightarrow (1+|\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n)$$

Denote $\langle\xi\rangle = (1+|\xi|^2)$

$$\int_{\mathbb{R}^n} |\hat{u}(\xi)| d\xi = \int_{\mathbb{R}^n} \langle\xi\rangle^{-s/2} \langle\xi\rangle^{s/2} |\hat{u}(\xi)| d\xi \leq \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{-s} d\xi\right)^{1/2} \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{s} |\hat{u}|^2 d\xi\right)^{1/2}$$

$1^\circ)$ By assumption $(u \in H^s(\mathbb{R}^n)) \Rightarrow \int \langle\xi\rangle^s |\hat{u}|^2 d\xi < \infty$

$2^\circ)$ $\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^s} d\xi = C \int_0^\infty \left(\frac{1}{1+r^2}\right)^{-s} r^{n-1} dr \leq C + \tilde{C} \int_1^\infty r^{-2s+n-1} dr$

$\leq \infty$ when $-2s+n-1 < -1 \Rightarrow s > n/2$

1° & 2° tell's that $\hat{u} \in L^1(\mathbb{R}^n)$

Fourier analysis say's if $u, \hat{u} \in L^1(\mathbb{R}^d)$ then we have an expression

$$u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \hat{u}(\xi) d\xi \quad \text{a.e } x \in \mathbb{R}^n$$

so $|u(x)| \lesssim C\|u\|_{H^s(\mathbb{R}^n)} \Rightarrow u \in L^\infty(\mathbb{R}^n)$

Also Fourier analysis tell's that:

$\mathcal{F}: L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$, so we have continuous representative for $\underline{u}$

HW 5 EX. 2

Trace-lause

Lause 1.18 Olkoon $s > 1/2$. Tällöin

Trace-kuvaus

$$T_0 : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^{n-1})$$

$$(T\phi)(x') = \phi(x', 0), \quad x' \in \mathbb{R}^{n-1}$$

laajenee jatkuvaksi kuvaukseksi

$$T_0 : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$$

Huomautus: T on surjektiivinen kuvaus [RR, 6.6.4]

Tod Olkoon $\phi \in C_0^\infty(\mathbb{R}^n)$, $g(x') = \phi(x', 0)$

Määritellään n:nen muuttujan suhteen Fourier-muunnos

$$\tilde{\phi}(x', \xi_n) = (\mathcal{F}_{x_n \to \xi_n} \phi)(x', \xi_n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix_n \cdot \xi_n} \phi(x', x_n)\, dx_n$$

Tällöin

$$g(x') = \phi(x', 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i0\cdot\xi_n} \tilde{\phi}(x', \xi_n)\, d\xi_n$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\phi}(x', \xi_n)\, d\xi_n.$$

HW 5 EX 2

Tällöin Fourier-muunnos toteuttaa

$$\hat{g}(\xi') = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \hat{\phi}(\xi', \xi_n)\, d\xi_n \quad , \xi' \in \mathbb{R}^{n-1}$$

$\mathcal{F}_{n-1} g =$ (n-1)-ulot. Fourier-muunnos

$\mathcal{F}_n \phi =$ n-ulotteinen F-muunnos

Ja saamme

$$\|g\|^2_{H^{s-1/2}(\mathbb{R}^{n-1})} \leq C_1 \int_{\mathbb{R}^{n-1}} (1+|\xi'|^2)^{s-1/2} |\hat{g}(\xi')|^2 d\xi'$$

$\hat{\phi} = \langle \xi \rangle^{-s/2} \cdot (\langle \xi \rangle^{s/2} \hat{\phi})$

$$\leq C_1 \int_{\mathbb{R}^{n-1}} (1+|\xi'|^2)^{s-1/2} \left| \int_{\mathbb{R}} \hat{\phi}(\xi', \xi_n)\, d\xi_n \right|^2 d\xi'$$

$$\underset{\text{Cauchy-Schwartz}}{\leq} C_1 \int_{\mathbb{R}^{n-1}} \Big[(1+|\xi'|^2)^{s-1/2} \left(\int_{\mathbb{R}} (1+|\xi'|^2+|\tilde{\eta}|^2)^{-s}\, d\tilde{\eta} \right) \cdot$$

$$\cdot \left(\int_{\mathbb{R}} (1+|\xi'|^2+|\eta|^2)^{s} |\hat{\phi}(\xi', \eta)|^2 d\eta \right) \Big] d\xi_n$$

HW 5 EX 2

Nyt muuttujanvaihdolla $\tilde{\eta} := (1+|\xi'|^2)^{1/2} y$ (ξ' on kiinteä)

$$\int_{\mathbb{R}} (1+|\xi'|^2+|\tilde{\eta}|^2)^{-s}\, d\tilde{\eta}$$

$$= \int_{\mathbb{R}} \left[(1+|\xi'|^2)(1+|y|^2)\right]^{-s} \cdot (1+|\xi'|^2)^{1/2}\, dy$$

$$= (1+|\xi'|^2)^{-s+1/2} \cdot \underbrace{\int_{\mathbb{R}} (1+|y|^2)^{-s}\, dy}_{=C_s<\infty \text{ koska } s>\frac{1}{2}.}$$

Siispä $g = T_0\phi$:lle pätee

$$\|T_0\phi\|^2_{H^{s-1/2}(\mathbb{R}^{n-1})}$$

$$\leq C_1 \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (1+|\xi'|^2)^{s-1/2} \cdot C_s(1+|\xi'|^2)^{-s+1/2} \cdot (1+|\xi'|^2+|\eta|^2)^{s} |\hat{\phi}(\xi',\eta)|^2\, d\xi' d\eta$$

$$\leq C_1 C_s \|\phi\|^2_{H^s(\mathbb{R}^n)}.$$

Väite seuraa, koska $C_0^\infty(\mathbb{R}^n) \subset H^s(\mathbb{R}^n)$ on tiheä osajoukko. □

HW 5. EX 3.

Exersise 4/14 Tell's that when U is bounded and $1 \le p \le \infty$ and $v \in W^{1,p}(U)$ then $|u| \in W^{1,p}(U)$

Then Because

$u^+ := \max\{u, 0\}$ we have also expression

$$u^+ = \frac{u + |u|}{2} \quad \text{a.e } x \in U$$

Because $u, |u| \in W^{1,p}(U)$ then this presentative of u^+ belongs Because it's linear combination to $W^{1,p}(U)$

In a same way you see that $u^- \in W^{1,p}(U)$

HW 5 EX. 4

Let $U \subset \mathbb{R}^n$ open and Bounded

Let us Have an operator

$$Lu := -\sum_{i,j}^{n} \partial_j (a^{ij} \partial_i u) + cu \quad , \text{where } a^{ij}, c \in L^\infty(U)$$

The BiLinear Form is Defined:

$$B : H_0^1(U) \times H_0^1(U) \to \mathbb{R}$$

$$B[u,v] = \int_U \left\{ \sum_{i,j} a^{ij} u_{x_i} v_{x_j} + cuv \right\} dx$$

The first Point:

$$|B[u,v]| \leq \sum_{i,j}^{n} \| a^{ij} \|_{L^\infty(U)} \int_U |Du||Dv| \, dx + \|c\|_{L^\infty(U)} \int_U |u||v| \, dx$$

$$\leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1}$$

The Second Point: Poincare inequality say's

$$\|u\|^2_{L^2(U)} \leq \tilde{C} \|Du\|^2_{L^2(U)}$$

$$\|u\|^2_{H_0^1(U)} = \int_U u^2 dx + \int_U |Du|^2 dx \leq (1+\tilde{C}) \|Du\|^2_{L^2(U)}$$

ellipticity gives u

$$B[u,u] = \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} u_{x_j} \, dx + \int_U cu^2 \geq \theta \|Du\|^2_{L^2(U)} + \int_U cu^2 dx$$

$$\geq \frac{\theta}{1+\tilde{C}} \|u\|^2_{H_0^1(U)} - \mu \|u\|^2_{L^2} \geq \left(\frac{\theta}{1+\tilde{C}} - \mu \right) \|u\|^2_{H_0^1(U)}$$

$$\Rightarrow \mu \in \left(0, \frac{\theta}{1+\tilde{C}} \right) \quad \text{and} \quad c(x) \geq -\mu \quad \text{a.e } x \in U$$

HW 5. EX 5

$U \subset \mathbb{R}^n$ open and $\Delta u > 0$.

Let's assume that we have a LOCAL MAXIMUM at point $x_0 \in U$

then $\partial_j u(x_0) = 0$ For all j and $\partial_j^2 u(x_0) \leq 0$, But in that situation causes that $\Delta u(x_0) \leq 0$ ↯

on the other hand:

Let $U = \mathbb{R}$

$u(x) = e^x + e^{-x}$

$u'(x) = e^x - e^{-x}$

$u''(x) = e^x + e^{-x} > 0 \quad \forall x \in U$

HAS a LOCAL MINIMU and ALSO GLOBAL MINIMU at $x=0$

HW 5 EX.6

THEOREM 3 IN EVANS Book 6.3 Regularity Chapter say's: $a^{ij}, b^i, c \in C^\infty(U)$ and $f \in C^\infty(U)$.

Suppose $u \in H^1(U)$ is a weak solution of the elliptic PDE $Lu = f$, in U then $u \in C^\infty(U)$