PDE II, HW 4, EX 1

$x = (x_1, \dots, x_n)$

$B_n(0,1) \subset \mathbb{R}^N$, $S = \{x \in B_n(0,1) : x_n = 0\}$

$U = B_n \setminus S$

Let us define: $u : B_n(0,1) \to \mathbb{R}$

$$u(x) = \begin{cases} 1, & x_n > 0 \\ 0, & x_n < 0 \end{cases} \Rightarrow \underline{\underline{u \in C^\infty(U)}}$$

$\Rightarrow |u(x)| < 1 \Rightarrow u \in L^\infty(U)$

$D^j u(x) = \partial_{x_j} u = 0 = \partial_{x_j} u \in L^\infty(U)$
$\forall j \in \{1, \dots, n\}$

$\Rightarrow u \in W^{1,\infty}(U)$

Let $x_k = (0, 0, \dots, 0, \frac{1}{k})$
$y_k = (0, 0, \dots, 0, -\frac{1}{k})$

$|u(x_k) - u(y_k)| = 1 \le C |x_k - y_k| = C \cdot \frac{2}{k}$

$\Rightarrow C \geq \frac{k}{2} \to \infty$, $k \to \infty$

$\Rightarrow$ u is not Lipschitz continuous on $\underline{\underline{U}}$

$$\int_{B(0,1)} |u(x)|^n dx = c_n \int_0^1 |\ln\ln(1+\tfrac{1}{r})|^n r^{n-1} dr$$

Integrand is bounded on interval $[\varepsilon, 1]$

The only problem might rise when $\varepsilon \to 0$

Let $1+\frac{1}{r} > e \Rightarrow r < \frac{1}{e-1}$

So $\forall r \in (0, \frac{1}{e-1})$ $\ln\ln(1+\frac{1}{r}) > 0$

$\Rightarrow 0 < \ln\ln(1+\frac{1}{r}) \leq \ln(\frac{1}{r})$

on the other hand L'Hopital:

$$\lim_{r\to 0+} \frac{(\ln(\frac{1}{r}))^n}{(\frac{1}{r})^{n-1}} = \lim_{r\to 0+} \frac{n(\ln(\frac{1}{r}))^{n-1}\left(\frac{-\frac{1}{r^2}}{\frac{1}{r}}\right)}{(n-1)(\frac{1}{r})^{n-2}(-\frac{1}{r^2})} = \lim \left(\frac{n}{n-1}\right)\left(\frac{\ln(\frac{1}{r})}{\frac{1}{r}}\right)^{n-1}$$

Because $\lim_{r\to 0+} \frac{n}{n-1}\left(\frac{\ln(\frac{1}{r})}{\frac{1}{r}}\right)^{n-1} = \frac{n}{n-1}(0)^{n-1} = 0$

$$\Rightarrow \lim_{r\to 0+} (\ln\ln(1+\tfrac{1}{r}))^n r^{n-1} \to 0$$

So we see that the integrand is bounded in $(0,1]$ $\Rightarrow$ $\underline{\underline{u \in L^n(B(0,1))}}$

let $x \in B(0,1)\setminus\{0\}$

$$\partial_{x_j} u(x) = \partial_{x_j}\left(\ln\ln\left(1+\frac{1}{|x|}\right)\right) = -\frac{\frac{x_j}{|x|}}{\ln\left(1+\frac{1}{|x|}\right)\left(|x|^2+|x|\right)}$$

if the weak D^j exist, it is a.e $x \in U$

$$\underline{\partial_{x_j} u(x) = D^j u(x)}$$

first we show that $\partial_{x_j} u \in L^n(B(0,1))$

$$\int_{B(0,1)} |\partial_{x_j} u|^n dx = \int_{B(0,1)} \left|\frac{\frac{x_j}{|x|}}{\ln\left(1+\frac{1}{|x|}\right)\left(|x|^2+|x|\right)}\right|^n dx \le \int_{B(0,1)} \frac{1}{\left|\ln\left(1+\frac{1}{|x|}\right)\left(|x|^2+|x|\right)\right|^n} dx$$

$$\le c_n \int_0^1 \frac{r^{n-1}}{\left|\ln\left(1+\frac{1}{r}\right)\left(r^2+r\right)\right|^n} dr \le \int_0^1 \frac{r^{n-1}}{\left|\ln\left(1+\frac{1}{r}\right)\left(r^2+r\right)\right|^n} dr$$

$$\le \int_0^{1/e-1} \frac{r^{n-1}}{\left|\ln\left(1+\frac{1}{r}\right)\left(r^2+r\right)\right|^n} dr + \underbrace{\int_{1/e-1}^{1} \frac{r^{n-1}}{\left|\ln\left(1+\frac{1}{r}\right)\left(r^2+r\right)\right|^n} dr}_{\le M < \infty}$$

$$\int_0^{1/e-1} \frac{r^{n-1}}{\left|\ln\left(1+\frac{1}{r}\right)\right|^n \left(r^2+r\right)^n} dr \le \int_0^{\frac{1}{e-1}} \frac{r^{n-1}}{\left(\ln\left(1+\frac{1}{r}\right)\right)^2 \left(r^2+r\right)^n} dr \quad \left(\ln\left(1+\frac{1}{r}\right) > 1 \ \forall x \in \left(0, \frac{1}{e-1}\right)\right)$$

also $\forall x \in \left(0, \frac{1}{e-1}\right)$ $\quad r^{n-1} \le r^2 + r \Rightarrow \frac{r^{n-1}}{(r^2+r)^n} \le \frac{1}{(r^2+r)}$

$$\Rightarrow \le \int_0^{\frac{1}{e-1}} \frac{1}{\left(\ln\left(1+\frac{1}{r}\right)\right)^2 \left(r^2+r\right)} dr = \int_0^{\frac{1}{e-1}} \frac{d\left(\left(\ln\left(1+\frac{1}{r}\right)\right)^{-1}\right)}{dr} dr < M < \infty$$

So $\partial_{x_j} u \in L^n(B(0,1))$

LAST thing is to show that the Definition For the weak Derivative HOLD's IN $B(0,1)$

let $\phi \in C_0^\infty(B(0,1))$ and $\varepsilon > 0$ s.t $B(0,\varepsilon) \in \mathbb{R}^N$
and n is "outward-pointing" unit Normal $n = (n_1, n_2, .. n_i, .. n_n)$, $|n| = 1$

$$\int_{B(0,1)\setminus B(0,\varepsilon)} u \frac{\partial \phi}{\partial x_i} dx = - \int_{B(0,1)\setminus B(0,\varepsilon)} \frac{\partial u}{\partial x_i}(x)\phi(x) + \int_{\partial B(0,\varepsilon)} u \phi n_i \, ds$$

Denote $U_\varepsilon = B(0,1) \setminus B(0,\varepsilon)$

Because $\chi_{\{U_\varepsilon\}}(x)\, u(x) \frac{\partial \phi}{\partial x_i}(x) \to u(x) \frac{\partial \phi}{\partial x_i}(x)$

a.e $x \in B(0,1)$

and

$$\left| \chi_{\{U_\varepsilon\}} u(x) \frac{\partial \phi}{\partial x_i}(x) \right| \le \left\|\frac{\partial \phi}{\partial x_i}\right\|_{L^\infty} |u(x)| \in L^n(B(0,1))$$

LDC say's

$$\int_{B(0,1)} u \frac{\partial \phi}{\partial x_i} dx = \lim_{\varepsilon \to 0} \int_{B(0,1)} \chi_{\{U_\varepsilon\}}(x)\, u(x) \frac{\partial \phi}{\partial x_i}(x)\, dx = \lim_{\varepsilon \to 0} \int_{B(0,1)\setminus B(0,\varepsilon)} u(x) \frac{\partial \phi}{\partial x_i}(x) dx$$

also same argument HOLD's

$$- \int_{B(0,1)} \frac{\partial u}{\partial x_i}(x)\phi(x) = \lim_{\varepsilon \to 0} \int_{B(0,1)\setminus B(0,\varepsilon)} \frac{\partial u}{\partial x_i}(x)\phi(x)\, dx$$

$$\left| \int_{\partial B(0,\varepsilon)} u \phi n_i \, ds \right| \le \int_{\partial B(0,\varepsilon)} |u \phi n_i| \, ds \le \|\phi\|_{L^\infty} |n| \int_{\partial B(0,\varepsilon)} |u| \, ds$$

So the might be in term

$$\int_{\partial B(0,\varepsilon)} |u| \, ds = \int_{\partial B(0,\varepsilon)} \left|\ln \ln\left(1+\frac{1}{|x|}\right)\right| ds = \left|\ln \ln\left(1+\frac{1}{|x|}\right)\right| \int_{\partial B(0,\varepsilon)} ds$$

$$= C_n \varepsilon^{n-1} \left|\ln \ln\left(1+\frac{1}{\varepsilon}\right)\right| = \frac{\ln \ln\left(1+\frac{1}{\varepsilon}\right)}{\left(\frac{1}{\varepsilon}\right)^{n-1}}$$, when ε is small enough

$$\lim_{\varepsilon \to 0} \frac{\ln \ln\left(1+\frac{1}{\varepsilon}\right)}{\left(\frac{1}{\varepsilon}\right)^{n-1}} \overset{\text{L'Hopital}}{=} \lim_{\varepsilon \to 0} \frac{\frac{\frac{-1}{\varepsilon^2}}{1+\frac{1}{\varepsilon}}}{\ln\left(1+\frac{1}{\varepsilon}\right)} \Big/ (n-1)\left(\frac{1}{\varepsilon}\right)^{n-2}\left(\frac{-1}{\varepsilon^2}\right) = \lim_{\varepsilon \to 0} \frac{\frac{1}{\ln\left(1+\frac{1}{\varepsilon}\right)\left(1+\frac{1}{\varepsilon}\right)}}{(n-1)\left(\frac{1}{\varepsilon}\right)^{n-2}}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\underbrace{(n-1)}_{\ge 1} \underbrace{\left(\frac{1}{\varepsilon}\right)^{n-2}}_{\ge 1} \underbrace{\ln\left(1+\frac{1}{\varepsilon}\right)}_{\to \infty} \underbrace{\left(1+\frac{1}{\varepsilon}\right)}_{\to \infty}} = \frac{1}{\infty} = 0$$

So we have $\phi \in C_0^\infty(B(0,1))$

$$\int_{B(0,1)} u \frac{\partial \phi}{\partial x_i} \, dx = - \int_{B(0,1)} \frac{\partial u}{\partial x_i} \phi \, dx$$

$$\Rightarrow \underline{u \in W^{1,n}(B(0,1))}$$

$F \in C^1(\mathbb{R})$ and $F' \in L^\infty(\mathbb{R})$

Let $\{u_j\} \subset C^\infty(U)$ s.t

$$\begin{cases} u_k \xrightarrow{L^p(U)} u \\ \partial_{x_j} u_k \xrightarrow{L^p(U)} \partial_{x_j} u \quad , \forall j \in \{1, \dots, n\} \end{cases}$$

$$\int_U |F \circ u_k(x) - F \circ u(x)|^p dx = \int_U |F(u_k(x)) - F(u(x))|^p dx$$

$$\leq \|F'\|_\infty \int_U |u_k(x) - u(x)|^p dx \to 0 \quad , k \to \infty$$

that is Because:

$$|F(x) - F(y)| = |F'(\xi)(x-y)| \leq \|F'\|_\infty |x-y| \quad , \forall x, y \in \mathbb{R}$$

$\Big($ also we see following : Let $x_0 \in U$ constant

$$|F \circ u(x)| \leq |F \circ u(x) - F \circ u(x_0)| + |F \circ u(x_0)|$$

$$\leq \|F'\|_\infty |u(x) - u(x_0)| + C \leq M < \infty \quad \forall x \in U \Big)$$

$\Rightarrow$ $\underline{\underline{F \circ u \ L^p(U)}}$ $\forall 1 \leq p \leq \infty$

AND $\underline{\underline{F \circ u_k \xrightarrow{L^p} F \circ u}}$

Lemma: $u, v \in L^p(U)$ and U is Bounded

then $v = D^\alpha u$ if and only if

there exist $\varphi_j \in C^\infty(U)$ s.t

$$\begin{cases} \varphi_j \to u & \text{in } L^p(U) \\ D^\alpha \varphi_j \to v & \text{in } L^p(U) \end{cases}$$

This is left as an easy exercise !!

So it's left to show that

there is $v = F'(u) D^\alpha u = F'(u)\partial_{x_j} u$ in $L^p(U)$

and $D^\alpha(v) = D^\alpha(F \circ u_k) = \partial_{x_j}(F \circ u_k)$ in $C^\infty(U)$ $\forall k \in \mathbb{N}$

s.t $\partial_{x_j}(F \circ u_k) \xrightarrow{L^p(U)} F'(u)\partial_{x_j} u$

$$\int_U |v|^p dx = \int_U |F'(u)\partial_{x_j} u|^p \le \|F'\|_\infty^p \int_U |\partial_{x_j} u|^p dx \le \|F'\|_\infty^p \|u\|_{W^{1,p}}^p$$

$\Rightarrow v \in L^p(U)$

$$\int_U |\partial_{x_j}(F \circ u_k) - F'(u)\partial_{x_j} u|^p dx = \int_U |\underbrace{\partial_x F(u_k)}_{=F'(u_k)} \partial_{x_j} u_k - F'(u)\partial_{x_j} u|^p dx$$

$$= \int_U |F'(u_k)\partial_{x_j} u_k - F'(u_k)\partial_{x_j} u + F'(u_k)\partial_{x_j} u - F'(u)\partial_{x_j} u|^p dx$$

$\Rightarrow$

$$\le 2^{p-1}\int_U |F'(u_k)-F'(u)|^p|\partial_{x_j}u|^p\,dx + 2^{p-1}\int_U |F'(u)|^p|\partial_{x_j}u_k-\partial_{x_j}u|^p dx$$

taking subsequence $u_k := u_{k_n}$ we know that $u_k \to u$ a.e $x\in U$

and F' is continuous then

$$F'\circ u_k \to F'\circ u \quad \text{a.e } x\in U$$

$$|F'(u_k)-F'(u)|^p|\partial_{x_j}u|^p \le \|F'\|^p_{L^\infty}|\partial_{x_j}u|^p \in L^p(U)$$

so we have majorant and can use LDC

so $2^{p-1}\int_U |F'(u_k)-F'(u)|^p|\partial_x u|^p dx \to 0$, $k\to\infty$

and $2^{p-1}\int_U |F'(u)|^p|\partial_{x_j}u_k-\partial_{x_j}u|^p dx \le C\|u_k-u\|^p_{W^{1,p}(U)} \to 0$

when $k\to\infty$

(If we belive Lemma then)

$$D^{x_j}(F\circ u) = F'(u)D^{x_j}u$$

CASE $p = \infty$

$U \subset \mathbb{R}^n$ bounded $\quad u \in L^\infty(U) \Rightarrow u \in L^q(U)$

, $q \in [1, \infty)$ and $\partial_{x_j} u \in L^\infty(U) \Rightarrow \partial_{x_j} u \in L^q(U)$

so the first part of proof tell's

$$\partial_{x_j}(v) = F'(u)\, \partial_{x_j} u \qquad \text{and}$$

$$|v(x)| = |F \circ u(x)| \leq |F \circ u(x) - F \circ u(x_0)| + \underbrace{|F \circ u(x_0)|}_{= M}$$

$$|F(u(x)) - F(u(x_0))| \leq |F'(\xi)|\, |u(x) - u(x_0)| \quad , \xi \in (u(x), u(x_0))$$

$$\leq \|F'\|_{L^\infty}\, 2 \|u\|_{L^\infty}$$

$$\Rightarrow F \circ u \in L^\infty(U)$$

$$|\partial_{x_j} v(x)| = |F'(u(x))\, \partial_{x_j} u(x)| \leq \|F'\|_{L^\infty} \|\partial_{x_j} u\|_{L^\infty}$$

$$\Rightarrow \partial_{x_j}(F \circ u) \in L^\infty(U)$$

$$\Rightarrow v = F \circ u \in W^{1,\infty}(U)$$

let $\varepsilon > 0$, U is bounded

Let us Define $F_\varepsilon(x) = \sqrt{x^2+\varepsilon^2} - \varepsilon$

we see that $F_\varepsilon \in C^1(\mathbb{R})$ and $0 \le F_\varepsilon(x) \le |x|$ $\forall x \in \mathbb{R}$

and $F_\varepsilon'(x) = \dfrac{x}{\sqrt{x^2+\varepsilon^2}} \Rightarrow |F_\varepsilon'(x)| \le 1$, $\forall x \in \mathbb{R}$

and $u \in W^{1,p}(U)$ then using the EX 3

we know that $F_\varepsilon \circ u \in W^{1,p}(U)$

and $D^{x_j}(F_\varepsilon \circ u) = F_\varepsilon'(u) D^{x_j} u$

$*$ and Def. For weak Der holds

$$\int_U D^{x_j}(F_\varepsilon \circ u)\varphi\, dx = -\int_U f_\varepsilon \circ u\, D^{x_j}\varphi\, dx$$

So $\cancel{\text{then}}$ $|F_\varepsilon \circ u| = \left|\sqrt{|u(x)|^2+\varepsilon^2} - \varepsilon\right| \le |u(x)|$ $\forall x \in U$

and $u \in L^p(U)$ and

$F_\varepsilon \circ u(x) \to |u(x)|$ pointwise . let $\phi \in C_0^\infty(U)$

and also $|u D^{x_j}\phi(x)| \le \|D^{x_j}\|_{L^\infty} |u(x)| \in L^1(U)$

By LDK

$$\lim_{\varepsilon \to 0} \int_U f_\varepsilon \circ u\, D^{x_j}\phi\, dx = \int_U |u| D^{x_j}\phi\, dx$$

then the left side of $*$

$$\int_U D^{x_j}(F_\varepsilon \circ u)\varphi\, dx = \int_U F_\varepsilon'(u) D^{x_j} u\, \varphi\, dx$$

$$= \int_U \frac{u(x)}{\sqrt{u(x)^2+\varepsilon^2}} D^{x_j} u \varphi \, dx$$

Because $\left| \frac{u(x)}{\sqrt{u(x)^2+\varepsilon^2}} D^{x_j} u \right| \leq |D^{x_j} u| \in L^1(U) \Rightarrow \|\varphi\|_{L^\infty} |D^{x_j} u| \in L^1(U)$

and $\frac{u(x)}{\sqrt{u(x)^2+\varepsilon^2}} \longrightarrow \operatorname{sgn}(u(x))$

LDC say's

$$\lim_{\varepsilon \to 0} \int_U \frac{u(x)}{\sqrt{u(x)^2+\varepsilon^2}} D^{x_j} u(x) \varphi \, dx = \int_U \operatorname{sgn}(u(x)) D^{x_j} u(x) \varphi \, dx$$

we get : $\forall \varphi \in C_0^\infty(U)$

$$\int_U |u| D^{x_j} \varphi \, dx = - \int_U \operatorname{sgn}(u(x)) D^{x_j} u(x) \varphi \cdot dx$$

So $D^{x_j}(|u|) = \operatorname{sgn}(u) D^{x_j} u \in \underline{L^p(U)}$ $\underline{\forall j \in \{1, \dots, n\}}$

$\Longrightarrow |u| \in W^{1,p}(U)$ when $u \in W^{1,p}(U)$

$$\Omega = [0,1]\times[0,1] \quad , \quad k \in C([0,1]\times[0,1])$$

$$K \in \mathcal{L}(L^2(\Omega)) \qquad Kf(x) = \int_0^1 k(x,s)\, f(s)\, ds \quad , \quad x \in [0,1]$$

is K compact operator ?

Let $n \in \mathbb{N}$

Let $\quad a_k = \frac{k}{n}$
$\qquad b_k = \frac{k}{n} \qquad k = 0, \dots, n-1$

$$\Omega_{k,l} = \left[\frac{k}{n}, \frac{k+1}{n}\right) \times \left[\frac{l}{n}, \frac{l+1}{n}\right) \subset [0,1]\times[0,1]$$

So $\Omega = \bigcup_{k,l=0}^{n-1} \Omega_{k,l}$ and choose a point

$$x_{k,l} = \left(\frac{k+\frac{1}{2}}{n}, \frac{l+\frac{1}{2}}{n}\right) \qquad k \in \{0,\dots,n-1\},\ l \in \{0,\dots,n-1\}$$

and Defin function $k : \Omega \to \mathbb{R}$

$$k^{(n)}(x,y) = \sum_{k,l=0}^{n-1} k(x_{k,l})\, \chi_{\Omega_{k,l}}(x,y)$$

$$K^{(n)} f(x) = \int_\Omega \sum_{k,l=0}^{n-1} k(x_{k,l})\, \chi_{\Omega_{k,l}}(x,y)\, f(y)\, dy$$

$$= \sum_{k,l=0}^{n-1} k(x_{k,l}) \int_\Omega \chi_{\Omega_{k,l}}(x,y)\, f(y)\, dy$$

$$= \sum_{k,l=0}^{n-1} k(x_{k,l}) \int_\Omega \chi_{[a_k, a_{k+1}]}(x)\, \chi_{[b_l, b_{l+1}]}(y)\, f(y)\, dy$$

$$K^n f(x) = \sum_{k,l=0}^{n-1} k(x_{kl})\, \chi_{[a_k, a_{k+1}]}(x) \int_{a_l}^{a_{l+1}} f(y)\,dy$$

So $K^n f \in \text{span}\left\{ \chi_{[0,\frac{1}{n})}, \chi_{[\frac{1}{n},\frac{2}{n})}, \dots, \chi_{[\frac{n-1}{n},1]} \right\}$

$\Rightarrow K^n : L^2[0,1] \to L^2[0,1]$ is compact

Schur's Lemma:

If $\sup_x \int_\Omega |K(x,y)|\,dy \,,\ \sup_y \int_\Omega K(x,y)\,dx \le M < \infty$

then $K : L^2(\Omega) \to L^2(\Omega)$ and $\|K\| \le M$

Proof: (This can be found in Petrus lecture notes: integral equations)

$$|Kf(x)| \le \int_\Omega |k(x,y)|\,|f(x)|\,dy$$

$$\le \Big(\underbrace{\int_\Omega \left(|k(x,y)|^{\frac{1}{2}}\right)^2}_{M^{\frac{1}{2}}} \Big)^{\frac{1}{2}} \left(\int_\Omega \left(|k(x,y)|^{\frac{1}{2}}\right)^2 |f(x)|^2\,dy \right)^{1/2}$$

$$\|Kf\|_{L^2}^2 = \int_\Omega |Kf(x)|^2\,dx \le M \int_{\Omega\times\Omega} |k(x,y)|\,|f(x)|^2\,dy\,dx$$

$$\le M \Big(\sup_y \int_\Omega |k(x,y)|\,dy \int_\Omega |f(x)|^2\,dx$$

$$\le M^2 \|f\|_{L^2}^2$$

Hw. 4. Ex 5 Page 3 | Let pick $(x,y) \in [0,1] \times [0,1] \Rightarrow (x,y) \in \Omega_{k,l}$

$$|K^n(x,y) - K(x,y)| = \left|K\left(\frac{k+\frac{1}{2}}{n}, \frac{l+\frac{1}{2}}{n}\right) - K(x,y)\right|$$

But K is uniformly continuous in $[0,1] \times [0,1]$

Let $\varepsilon > 0$ $\exists\, n_\varepsilon$ s.t.

$$|K(x,y) - K(\tilde{x},\tilde{y})| < \varepsilon$$

when $|(x,y) - (\tilde{x},\tilde{y})| < \frac{1}{n_\varepsilon}$

So we see $\forall\, n > 2n_\varepsilon$

$$\sup_{(x,y) \in [0,1]\times[0,1]} |K^n(x,y) - K(x,y)| = \sup_{(x,y) \in [0,1]\times[0,1]} \left|K\left(\frac{k+\frac{1}{2}}{n}, \frac{l+\frac{1}{2}}{n}\right) - K(x,y)\right| \leq \varepsilon$$

So $\displaystyle\int_{0}^{1} |K^n - K(x,y)|\, dy \leq \varepsilon \int_\Omega dy = \varepsilon$

and $\displaystyle\int_0^1 |(K^n - K)(x,y)|\, dx \leq \varepsilon$

$\Big\}\ \forall\, n > 2n_\varepsilon$

So Schur's Lemma:

$$K^n - K : L^2[0,1] \to L^2[0,1]$$

and $\|K^n - K\| \leq \varepsilon$

$\begin{cases} K^n \text{ are compact} \\ K^n \to K \text{ in } \mathcal{L}(L^2[0,1]) \end{cases} \Longrightarrow$ $\underline{K \text{ is compact}}$

HW 4. EX 6

Look EVANS for example in Appendix D remark:

let $K: H \to H$ compact then:

$\begin{cases} \text{for each } g \in H, \ f + Kf = g \\ \text{has a unique solution} \end{cases}$

or else

$\begin{cases} \text{the homogeneous equation } f + Kf = 0 \\ \text{has non trivial solution } f \neq 0 \end{cases}$