PDE II HW2. EX. 1.

Let $V, U \Subset \mathbb{R}^N$, $\overline{V}$ is compact

then $\exists\, \varepsilon_0 > 0$ s.t $\operatorname{dist}(\overline{V}, \partial U) > 4\varepsilon_0$

Let $W = \{x \in U : \operatorname{Dist}(x, V) < \varepsilon_0\}$

Let $\phi_\varepsilon \in C_0^\infty(B(0,\varepsilon_0))$ s.t $\int_{B(0,\varepsilon_0)} \phi_\varepsilon(x)\,dx = 1$

let $u_0 = \begin{cases} 1, & x \in W \\ 0, & x \in \mathbb{R}^N \setminus W \end{cases}$

Let's Define $\hat{U} := u_0 * \phi_\varepsilon$

we know that $\hat{U} \in C^\infty(\mathbb{R}^N)$

also if $x \in V$ we see that

$$\hat{U}(x) = \int_U u_0(y)\,\phi_\varepsilon(x-y)\,dy = \int_{B(x,\varepsilon)} \phi_\varepsilon(x-y)\,dy = 1$$

also if $x \in U$ s.t $\operatorname{Dist}(x, \overline{W}) \geq 3\varepsilon_0$

$\hat{U}(x) = 0$ then we DEFINE

$$\underline{U = \hat{U}|_U}$$

PDE II HW 2. EX. 2

$U \subset \mathbb{R}^d$ $\bar{U}$ compact and

$\bar{U} \subset \bigcup_{i=1}^{N} V_i$ is open, finite cover,

Let us take $\varepsilon_0 > 0$ s.t

$$\mathrm{Dist}\left(\bar{U}, \partial\left(\bigcup_{i=1}^{N} V_i\right)\right) > 4\varepsilon_0$$

Let us define set's

$$E_j = \{x \in \bar{U} : \mathrm{Dist}(x, \mathbb{R}^n \setminus V_j) \geq 2\varepsilon_0\}$$

So we see that E_j is a compact and $E_j \subset V_j$

and $\mathrm{Dist}(E_j, \partial V_j) \geq 2\varepsilon_0$

and $\bigcup_{j=1}^{n} E_j = \bar{U}$

<u>using EX. 1 we can find</u>

$\varphi_j \in C_0^\infty(V_j)$ s.t $\varphi_j(x) = 1$, when $x \in E_j$

and $0 \leq \varphi_j \leq 1$

Let's Define $\varphi(x) = \sum_{j=1}^{N} \varphi_j(x)$

we see that $\varphi \in C_0^\infty\left(\bigcup_{i=1}^{N} V_i\right)$

Let's Define $x \in U$

$$\xi_j(x) := \frac{\varphi_j(x)}{\varphi(x)} \in C_0^\infty(V_i)$$

when $x \in U$ $\Rightarrow x \in E_j \Rightarrow$ $1 \leq \varphi(x) \leq N$

so that $\sum_{i=1}^{N} \xi_j(x) = \frac{\sum \varphi_j(x)}{\varphi(x)} = 1 \, ; \forall x \in U$

PDE II HW 2 EX. 3

$u \in W^{1,p}(0,1)$ the $\exists \{u_n\}$, $u_n \in C^\infty[0,1]$

and $u_n \to u$ $W^{1,p}(0,1)$

So we have also that

$$u_n \xrightarrow{L^p} u$$
$$u_n' \xrightarrow{L^p} u'$$

where u' is the weak-derivative of u

Let's take subsequence so that we have $(u_{n_{k_j}} := u_j)$

$$\begin{cases} u_j(x) \to u(x) & \text{a.e } x \in (0,1) \\ u_j'(x) \to u'(x) & \text{a.e } x \in (0,1) \end{cases}$$

Let's pick a point $x_0 \in (0,1)$

So, that $u_j(x_0) \to u(x_0)$

AND $u_j(x) = u_j(x_0) + \int_{x_0}^{x} u_j'(y)\,dy$

$$\lim_{j\to\infty} u_j(x) = \lim_{j\to\infty} u_j(x) + \lim_{j\to\infty} \int_{x_0}^{x} u_j'(y)\,dy$$

$$= u(x_0) + \lim_{j\to\infty} \int_{x_0}^{x} u_j'(y)\,dy$$

and

$$\left| \int_{x_0}^{x} u_j'(y)\,dy - \int_{x_0}^{x} u'(y)\,dy \right| \leq \int_{x_0}^{x} |u_j'(y) - u'(y)|\,dy$$

$$\leq C \left(\int_{x_0}^{x} |u_j'(y) - u'(y)|^p \right)^{1/p}$$

$$\leq C \|u_j' - u'\|_{L^2} \to 0$$

So we have $\lim_{j\to\infty} \int_{x_0}^{x} u_j'(y)\,dy = \int_{x_0}^{x} u'(y)\,dy$

We have now a.e $x \in (0,1)$

$$u(x) = \lim_{j\to\infty} u_j(x) = u(x_0) + \int_{x_0}^{x} u'(y)\,dy$$

So $u'(y) \in L^1(0,1)$ and u' is defined a.e $x \in (0,1)$

and $\hat{u}(x) = u(x_0) + \int_{x_0}^{x} u'(y)\,dy$ $x \in (0,1)$

gives us that $\hat{u}$ is absolutely-cont..

$u = \hat{u}$ a.e $x \in (0,1)$

4/ Because for a.e $x, y \in (0,1)$

$$|u(x) - u(y)| = \left| u(x_0) + \int_{x_0}^{x} Du(\xi)\,d\xi - u(x_0) - \int_{x_0}^{y} Du(\xi)\,d\xi \right|$$

$$= \left| \int_{x_0}^{x} Du(\xi)\,d\xi - \int_{x_0}^{y} Du(\xi)\,d\xi \right|$$

$$\leq \int_{y}^{x} |Du(\xi)|\,d\xi \leq \left(\int_{y}^{x} d\xi \right)^{\frac{1}{q}} \left(\int_{y}^{x} |Du(\xi)|^p\,d\xi \right)^{1/p}$$

$$\leq |x-y|^{\frac{1}{q}} \|Du\|_{L^p(0,1)}$$

PDE HW2 . EX 4

PDE II HW. 2 EX 5

Look APPENDIX D.5 ~~[illegible]~~

PDE II HW. 2 EX 6

Theorem 1 (Rellich-Kondrachov compactness theorem)

$W^{1,p}(U) \subset\subset L^q(U)$ for each $1 \le q < p^*$

when $1 \le p < n$, $p^* = \frac{np}{n-p}$

and so we have that $1 \le p < p^*$

$W^{1,p}(U) \subset\subset L^p(U)$