PDE II HW 1 Ex. 1

$u(x) = |x|^{1/2}$, $\Omega = (-1,1)$

$$\|u\|_{C^{0,\gamma}(\bar\Omega)} = \|u\|_{C(\bar\Omega)} + [u]_{C^{0,\gamma}(\bar\Omega)}$$

$$\|u\|_{C(\bar\Omega)} := \sup_{x\in\Omega} |u(x)| = 1$$

So we see : $\exists \gamma \in (0,1]$, $\exists C > 0$.

$u \in C^{0,\gamma}(\bar\Omega)$ if $[u]_{C^{0,\gamma}(\bar\Omega)} \leq C$, $C \in \mathbb{R}$

Let $x,y \in (-1,1)$, $x \neq y$

$$\frac{|u(x)-u(y)|}{|x-y|^\gamma} = \frac{\left||x|^{1/2}-|y|^{1/2}\right|}{|x-y|^\gamma} = \frac{\left||x|^{1/2}-|y|^{1/2}\right|\left||x|^{1/2}+|y|^{1/2}\right|}{|x-y|^\gamma \left||x|^{1/2}+|y|^{1/2}\right|}$$

$$= \frac{\left||x|-|y|\right|}{|x-y|^\gamma \left||x|^{1/2}+|y|\right|^{1/2}} = \frac{\left||x|-|y|\right|^{1/2}\left||x|-|y|\right|^{1/2}}{|x-y|^\gamma \left||x|^{1/2}+|y|^{1/2}\right|} \leq \frac{\left||x|-|y|\right|^{1/2}\left||x|+|y|\right|^{1/2}}{|x-y|^\gamma \left||x|+|y|\right|^{1/2}}$$

$$\leq \frac{\left||x|-|y|\right|^{1/2}}{|x-y|^\gamma} \leq \frac{|x-y|^{1/2}}{|x-y|^\gamma} = |x-y|^{\frac{1}{2}-\gamma}$$

So we see that $\forall x,y \in (-1,1)$, $x \neq y$

Let $\gamma \in [0, \frac{1}{2}]$ then

$$\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma} \leq 2^{\frac{1}{2}-\gamma} \leq \sqrt{2} \Rightarrow \underline{u \in C^{0,\gamma}(\bar\Omega)}$$

PDE II HW 1 EX. 1

If $\gamma \in (\frac{1}{2}, 1)$ choose $y = 0$, $x > 0$

then $\gamma = \frac{1}{2} + \varepsilon$

$$\frac{|\sqrt{x} - \sqrt{0}|}{|x - 0|^{\gamma}} = \frac{\sqrt{x}}{x^{\frac{1}{2}+\varepsilon}} = \frac{1}{x^{\varepsilon}} \to \infty, \quad x \to 0$$

so we see that $u \notin C^{0,\gamma}(\bar{\Omega})$ when $\gamma \in (\frac{1}{2}, 1]$

①

PDE II
HW 1 EX. 2

2) $C^{0,\gamma}(\bar U)$ is a vector-space

a) 1° $\forall u, v \in C^{0,\gamma}(\bar U)$: $u, v : U \to \mathbb{R}$

$\Rightarrow \exists\, C_1, C_2$ s.t

$$|u(x)-u(y)| \le C_1 |x-y|^\gamma \qquad \forall x,y \in U$$
$$|v(x)-v(y)| \le C_2 |x-y|^\gamma$$

$\Rightarrow |w(x)+w(y)| < (C_1+C_2)|x-y|^\gamma$, where $w := u+v$

$\Rightarrow u+v \in C^{0,\gamma}(\bar U)$

2 let $u \in C^{0,\gamma}(\bar U)$, $\alpha \in \mathbb{R}$
Let $w = \alpha u$

$\Rightarrow |w(x)-w(y)| \le |\alpha| C_1 |x-y|^\gamma$, $\forall x, y \in U$

$\Rightarrow \alpha u \in C^{0,\gamma}(\tilde U) \Rightarrow C^{0,\gamma}(\tilde U)$ is vector-space

where vector addition and multiplication by scalars are defined in a usual way.

---

b) $\|\cdot\|_{C^{0,\gamma}}$ is a norm

$$\|u\|_{C^{0,\gamma}(\bar U)} := \|u\|_{C(\bar U)} + [u]_{C^{0,\gamma}(\bar U)} ,$$

where $\|u\|_{C(\bar U)} := \sup\limits_{x \in U} |u(x)|$ and

$$[u]_{C^{0,\gamma}(\bar U)} := \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\gamma} \right\}$$

Defines a norm to the $C^{0,\gamma}(\tilde U)$

$\|\cdot\| : U \to \mathbb{R}+$ Positivity ok.

1° $u = 0 \Rightarrow \|u\|_{C^{0,\gamma}(\bar U)} = 0$

$\|u\|_{C^{0,\gamma}(\bar U)} = 0 \Rightarrow \|u\|_{C(\bar U)} = 0$ & $[u]_{C^{0,\gamma}(\bar U)} = 0$

so $\|u\|_{C(\bar U)} = 0 \Rightarrow u = 0$

2° Let $\alpha \in \mathbb{R}$, $u \in C^{0,\gamma}(\bar{U})$

$$\|\alpha u\|_{C^{0,\gamma}(\bar{U})} = \|\alpha u\|_{C(\bar{U})} + [\alpha u]_{C^{0,\gamma}(\bar{U})} = |\alpha| \, \|u\|_{C(\bar{U})} + |\alpha| [u]_{C^{0,\gamma}}$$

$$= |\alpha| \, \|u\|_{C^{0,\gamma}(\bar{U})}$$

$u, v \in C^{0,\gamma}(\bar{U})$

3° $\|u+v\|_{C^{0,\gamma}(\bar{U})} = \|u+v\|_{C(\bar{U})} + [u+v]_{C^{0,\gamma}(\bar{U})}$

a) $\|u+v\|_{C(\bar{U})} \leq \|u\|_{C(\bar{U})} + \|v\|_{C(\bar{U})}$

b) $$\frac{|u(x)+v(x)-u(y)-v(y)|}{|x-y|^\gamma} \leq \frac{|u(x)-u(y)|}{|x-y|^\gamma} + \frac{|v(x)-v(y)|}{|x-y|^\gamma}$$

$\forall x, y \in U$, $x \neq y$

$$\sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u(x)+v(x)-u(y)-v(y)|}{|x-y|^\gamma} \right\} \leq \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\gamma} + \frac{|v(x)-v(y)|}{|x-y|^\gamma} \right\}$$

$$\Rightarrow \qquad \| \qquad \leq \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\gamma} \right\} + \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|v(x)-v(y)|}{|x-y|^\gamma} \right\}$$

So $[u+v]_{C^{0,\gamma}(\bar{U})} \leq [u]_{C^{0,\gamma}(\bar{U})} + [v]_{C^{0,\gamma}(\bar{U})}$

a & b togeher gives us

$$\|u+v\|_{C^{0,\gamma}(\bar{U})} \leq \|u\|_{C^{0,\gamma}(\bar{U})} + \|v\|_{C^{0,\gamma}(\bar{U})}$$

2C. $\left(C^{0,\gamma}(\bar{U}), \|\cdot\|_{C^{0,\gamma}}\right)$ is complete. (3)

Let $\{u_n\}$ be cauchy

Let $\varepsilon > 0$ then

$$\|u_n - u_m\|_{C^{0,\gamma}(\bar{U})} \le \varepsilon \qquad \forall\, n,m \geq N_\varepsilon$$

$$\Rightarrow \|u_n - u_m\|_{C(\bar{U})} \le C \qquad \forall\, n,m \geq N_\varepsilon$$

$\Rightarrow \{u_n\}$ cauchy in $\left(C(\bar{U}), \|\cdot\|_\infty\right)$

$\Rightarrow \exists\, u$ s.t. $\|u_n - u\|_\infty \to 0$, $n \to \infty$

$$\sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u_n(x) - u_m(x) - u_n(y) + u_m(y)|}{|x-y|^\gamma} \right\} \le \varepsilon \qquad \forall\, n,m \geq N_\varepsilon$$

$$\Rightarrow \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u_n(x) - u_n(y)|}{|x-y|^\gamma} \right\} \le \varepsilon + \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|u_m(x) - u_m(y)|}{|x-y|^\gamma}$$

$$\Rightarrow \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u_n(x) - u_n(y)|}{|x-y|^\gamma} \right\} \le \varepsilon + C \qquad \forall\, n \geq N_\varepsilon$$

$$\cancel{\Rightarrow \lim_{n\to\infty} \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u_n(x) - u_n(y)|}{|x-y|^\gamma} \right\}}$$

$$\sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\gamma} \right\} = \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|\lim_{n\to\infty} u_n(x) - \lim u_n(y)|}{|x-y|^\alpha} \right. = \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \lim_{n\to\infty} \frac{|u_n(x) - u_n(y)|}{|x-y|^\alpha} \right.$$

4

But

$$\sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \lim_{n\to\infty} \frac{|u_n(x)-u_n(y)|}{|x-y|^\gamma} \right\} \leq \sup_{\substack{x,y\in U\\ x\neq y\\ n\geq N_\varepsilon}} \left\{ \frac{|u_n(x)-u_n(y)|}{|x-y|^\gamma} \right\} \leq \varepsilon + C$$

$$\Rightarrow \quad u \in C^{0,\gamma}(U)$$

Also : $\forall x,y \in U$, $x \neq y$

$$\frac{|u(x)-u_N(x)-u(y)+u_N(y)|}{|x-y|^\gamma} =$$

$$\lim_{n\to\infty} \frac{|u_n(x)-u_N(x)-u_n(y)+u_N(y)|}{|x-y|^\gamma}$$

On the other way ..

$$\frac{|u_n(x)-u_N(x)-u_n(y)+u_N(y)|}{|x-y|^\gamma} \leq \|u_n-u_N\|_{C^{0,\gamma}} \leq \varepsilon$$

$$\forall N, n \geq N_\varepsilon$$

So

$$\sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u(x)-u_N(x)-u(y)+u_N(y)|}{|x-y|^\gamma} \right\} \leq \varepsilon$$

we see that $u_n \to u$ in $C^{0,\gamma}(\bar{U})$

PPE II HW 1. EX. 3

$U \Subset \mathbb{R}^N \quad , \quad u \in L^1_{loc}(U)$

Let's think that we have two weak α-derivatives so that $D^\alpha u = w_1 \;,\; D^\alpha u = w_2 \;,\; w_1, w_2 \in L^1_{loc}(U)$

and we have $\forall \phi \in C_0^\infty(U)$

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U w_1 \phi \, dx = (-1)^{|\alpha|} \int_U w_2 \phi \, dx$$

$$\Rightarrow \quad \int_U (w_1 - w_2) \phi \, dx = 0 \quad , \quad \forall \phi \in C_0^\infty(U)$$

Let's define $w := w_1 - w_2$

let us choose some point $\hat{x} \in U$

AND $\varphi \in C_0^\infty(B(0,1)) \quad , \quad \int_{B(0,1)} \varphi \, dx = 1$

then $\varphi_\varepsilon(x) := \varepsilon^{-n} \varphi\left(\frac{x}{\varepsilon}\right) \quad , \quad \int_{B(0,\varepsilon)} \varphi_\varepsilon \, dx = 1 \quad , \quad \varphi_\varepsilon \in C_0^\infty(B(0,\varepsilon))$

when ε is small enough $\mathrm{Supp}(\varphi_\varepsilon(\hat{x} - y)) \subset U$

and we can have

$$w(\hat{x}) = \int_U w(\hat{x}) \varphi_\varepsilon(\hat{x} - y) \, dy$$

An we can write following Ex. 3

$$w(\hat{x}) = \int_U \big(w(\hat{x}) - w(y)\big)\,\varphi_\varepsilon(\hat{x}-y)\,dy + \int_U w(y)\,\varphi_\varepsilon(\hat{x}-y)\,dy$$

We know that

$$\int_U w(y)\,\varphi_\varepsilon(\hat{x}-y)\,dy = 0$$

and we have

$$w(\hat{x}) = \int_U \big(w(\hat{x}) - w(y)\big)\,\varepsilon^{-n}\,\varphi\left(\frac{\hat{x}-y}{\varepsilon}\right)dy = \varepsilon^{-n}\int_{B(\hat{x},\varepsilon)} \big(w(\hat{x}) - w(y)\big)\,\varphi\left(\frac{\hat{x}-y}{\varepsilon}\right)dy$$

$$\Rightarrow |w(\hat{x})| \le \|\varphi\|_\infty\,\varepsilon^{-n}\int_{B(\hat{x},\varepsilon)} |w(\hat{x}) - w(y)|\,dy$$

and using the Lebesgue Differentiation theorem

We have $\lim\limits_{\varepsilon\to 0} \varepsilon^{-n}\int_{B(\hat{x},\varepsilon)} |w(\hat{x}) - w(y)|\,dy \to 0$

a.e $\hat{x} \in U$ so

$w(\hat{x}) = 0$ a.e $\hat{x} \in U$

PDE II HW 1 . EX. 4

Let $0<\beta<\gamma\leq 1$

$$\frac{|u(x)-u(y)|}{|x-y|^\gamma} = \frac{|u(x)-u(y)|^{r_1}}{|x-y|^{s_1}} \frac{|u(x)-u(y)|^{r_2}}{|x-y|^{s_2}}$$

$$= \left(\frac{|u(x)-u(y)|}{|x-y|^{s_1/r_1}}\right)^{r_1} \left(\frac{|u(x)-u(y)|}{|x-y|^{s_2/r_2}}\right)^{r_2}$$

Let us choose $\frac{s_1}{r_1} = \beta$, $\frac{s_2}{r_2} = 1$

also must have $r_1 + r_2 = 1$ and $s_1 + s_2 = \gamma$

we see that

$$r_1 = \frac{r_1(1-s_1-s_2)}{1-s_1-s_2} = \frac{r_1(1-s_1-s_2)}{1-r_2-s_1} = \frac{r_1(1-s_1-s_2)}{r_1-s_1}$$

$$r_1 = \frac{r_1(1-s_1-s_2)}{r_1\left(1-\frac{s_1}{r_1}\right)} = \frac{1-\gamma}{1-\beta}$$

$$r_2 = 1-r_1 = \frac{1-\beta}{1-\beta} - \frac{1-\gamma}{1-\beta} = \frac{\gamma-\beta}{1-\beta}$$

So we get:

$$[u]_{C^{0,\gamma}(U)} \leq [u]_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} [u]_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}$$

Ex. 4

Let us write

$$[u]_{C^{0,\beta}(U)} = a_\beta \qquad \frac{1}{p} = \frac{1-\gamma}{1-\beta}$$

$$[u]_{C^{0,1}(U)} = a_1 \qquad \frac{1}{q} = \frac{\gamma-\beta}{1-\beta}$$

$$[u]_{C^{0,\gamma}(U)} = a_\gamma$$

$$\sup_{x\in U} |u(x)| = a_0$$

We can use Discrete version of Hölder:

$$\|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}} = (a_0 + a_\beta)^{\frac{1}{p}} (a_0 + a_1)^{1/q}$$

$$= \left(|a_0^{1/p}|^p + |a_\beta^{1/p}|^p \right)^{1/p} \left(|a_0^{1/q}|^q + |a_1^{1/q}|^q \right)^{1/q}$$

$$\geq \left| a_0^{1/p} a_0^{1/q} + a_\beta^{1/p} a_1^{1/q} \right| = a_0 + a_\beta^{1/p} a_1^{1/q}$$

$$= \|u\|_{C(U)} + [u]_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} [u]_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}$$

$$\geq \|u\|_{C(U)} + [u]_{C^{0,\gamma}(U)} = \|u\|_{C^{0,\gamma}(U)}$$

PDE-II HW 1. EX. 5

Theorem 2 gives us GLOBAL approximation where as Theorem 1 Tales abouth LOCAL approximation. see more from Textbook on lecture notes!

PDE-II HW 2. EX. 6

Look Definitions From Appendix C of the Text Book Dealing with C^1-BOUNDARY.

... AND see How That is Done in Theorem 3 in EVANS Book ....