Partial Differential Equations II

HW 6

Return to Jussi by Monday, November 24.

1. Let $U \subset \mathbb{R}^n$ be a bounded C^2–domain. We say that $u \in H_0^2(U)$ is a weak solution of the following boundary value problem for the biharmonic operator,

$$\Delta^2 u = 0 \quad \text{in } U,$$

$$u = \partial u/\partial \nu = 0 \quad \text{on } \partial U,$$

if

$$\int_U \Delta u \Delta v \, dx = \int_U f v \, dx$$

for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove that there always exists a unique weak solution of the above problem.

2. Assume that U is a bounded and connected C^1–domain and $f \in L^2(U)$. A function $u \in H^1(U)$ is a weak solution of the *Neumann problem*

$$-\Delta u = f \quad \text{in } U, \quad \partial_\nu u = 0 \quad \text{on } \partial U,$$

if

$$\int_U \nabla u \cdot \nabla v \, dx = \int_U f v \, dx$$

for all $v \in H^1(U)$. Prove that this problem has a weak solution if and only if

$$\int_U f \, dx = 0.$$

3. Assume that $u \in H^1(\mathbb{R}^n)$ has compact support, and assume that it is a weak solution of the *semilinear* equation

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is a smooth function with $c(0) = 0$ and $c' \geq 0$. Prove that $u \in H^2(\mathbb{R}^n)$.

4. Assume that U is a connected and bounded C^1–domain. Using a) energy methods and b) Maximum principles, give two different proofs showing that the only smooth solutions of the homogeneous Neumann problem

$$-\Delta u = 0 \text{ in } U,\ \partial_\nu u = 0 \text{ on } \partial U$$

are the constants.

Before trying the next questions, read section 6.5 in Evans. Do not worry too much about all the details in the proofs, we will worry about those in the lectures. However, try to form a general picture of the material covered.

5. Assume that L is a symmetric elliptic operator in a bounded domain. What is the dimension of the eigenspace corresponding to the principal eigenvalue?

6. To see how section 6.5 relates to an *inverse problem* read the wikipedia article *Hearing the shape of a drum.* Besides knowing the eigenfrequencies, can you think of any other information that is directly observable on the boundary that could be used?