Partial Differential Equations II

HW 5

Return to Jussi by Monday, November 10.

1. Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in L^\infty(\mathbb{R}^n)$ with bound
$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)},$$
with the constant C depending only on s and n. What can you say about the continuity of u?

2. Prove that if $n > 1$ and $s > 1/2$, then the function
$$\mathbb{C}_0^\infty(\mathbb{R}^n) \ni u \mapsto u|_{x_n=0} \in C_0^\infty(\mathbb{R}^{n-1})$$
has an extension to a bounded linear operator $H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1})$.

3. Assume that U is bounded, $1 \leq p \leq \infty$, and $u \in W^{1,p}(U)$. Prove, that $u^\pm \in W^{1,p}(U)$. Here $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$.

4. Let
$$Lu = -\Sigma_{i,j=1}^n \partial_j(a^{ij}\partial_i u) + cu$$
be a uniformly elliptic operator in the domain $U \subset \mathbb{R}^n$. Prove, that there exists a positive constant μ such that the bilinear form determined by L satisfies the hypotheses of the Lax-Milgram Theorem provided $c(x) \geq -\mu$ in U.

   Before trying the next questions, read sections 6.3 and 6.4 in Evans. Do not worry too much about all the details in the proofs, we will worry about those in the lectures. However, try to form a general picture of the material covered.

5. Assume that $\Delta u > 0$ in the domain U. What can you say about the location of local maxima and minima of u? **Hint**: this is simple, just remember that a function of one real variable has a negative second derivative at a strict local maxima.

6. Let L be an uniformly elliptic operator in a bounded domain that has C^∞ coefficients. What can you say about the interior regluarity of u satisfying
$$Lu = f \in C^\infty(U)?$$