Partial Differential Equations II

HW 4

Return to Jussi by Monday, October 20.

1. Give an example of an open set $U \subset \mathbb{R}^n$ and a function $u \in W^{1,\infty}(U)$ such that u is not Lipschitz continuous on U.

2. Prove that if $n > 1$, then the function
$$u(x) = \ln \ln \left(1 + \frac{1}{|x|}\right), \ |x| < 1,$$
belongs to $W^{1,n}(B(0,1))$. Here $B(0,1)$ is the open unit ball of R^n.

3. Assume that $F : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, and that F' is bounded. Suppose that $U \subset \mathbb{R}^n$ is bounded, and $u \in W^{1,p}(U)$ for some $1 \leq p \leq \infty$ and that it is real valued. Define $v = F(u)$. Prove that $v \in W^{1,p}(U)$ and that $D_{x_i} v = F'(u) D_{x_i} u$.

4. Assume that U is bounded, $1 \leq p \leq \infty$ and that $u \in W^{1,p}(U)$. Prove that $|u| \in W^{1,p}(U)$.

   Before trying the next questions, read section 6.2 in Evans again. Do not worry too much about all the details in the proofs, we will worry about those in the lectures. However, try to form a general picture of the material covered.

5. Fill in the details of the proof given in the lectures showing that an integral operator with a continuous kernel $k \in C[0,1]^2$ defines a compact operator on $L^2[0,1]$.

6. Let k be as above, and consider the integral operator
$$Kf(x) = \int_0^1 k(x,s)\, f(s)\, ds, \ 0 \leq x \leq 1.$$
Explain what the Fredholm Alternative says about the solvability of the integral equation
$$f + Kf = g, \ g \in L^2[0,1]$$
in $L^2[0,1]$.