Partial Differential Equations II

HW 3

Return to Jussi by Monday, October 6.

1. Let U be the open square $(-1,1) \times (-1,1)$. Define

$$u(x) = \begin{cases} 1 - x_1, \text{ if } x_1 > 0,\ |x_2| < x_1 \\ 1 + x_1, \text{ if } x_1 < 0,\ |x_2| < -x_1 \\ 1 - x_2, \text{ if } x_2 > 0,\ |x_1| < x_2 \\ 1 + x_2, \text{ if } x_2 > 0,\ |x_1| < -x_2 \end{cases}$$

   For which $1 \leq p \leq \infty$ does u belong to $W^{1,p}(U)$?

2. Let $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ be the gradient of u and $D^2 u = (\partial_{x_i x_j} u)_{i,j=1}^n$ be the Hessian of u. Let

$$|Du|^2 = \sum_{i=1}^{n} |\partial_{x_i} u|^2$$

   and

$$|D^2 u| = \sum_{i,j=1}^{n} |\partial_{x_i x_j} u|^2$$

   be their norms. Prove an interpolation inequality

$$\int_U |Du|^2 \, dx \leq C \left(\int_U u^2 \, dx \right)^{1/2} \left(\int_U |D^2 u|^2 \, dx \right)^{1/2}$$

   for all $u \in C_0^\infty(U)$. Assuming that ∂U is smooth, prove this inequality if $u \in H^2(U) \cap H_0^1(U)$.

3. Integrate by parts to prove that

$$\int_U |Du|^p \, dx \leq C \left(\int_U u^p \, dx \right)^{1/2} \left(\int_U |D^2 u|^p \, dx \right)^{1/2}$$

   for all $2 \leq p < \infty$ and all $u \in W^{2,p}(U) \cap W_0^{1,p}(U)$. **Hint**: Write $|Du|^p = \sum_i u_{x_i}^2 |Du|^{p-2}$.

4. Assume that U is connected and $u \in W^{1,p}(U)$, $Du = 0$ a.e. in U. Prove that u is constant a.e in U.

   Before trying the next questions, read sections 6.1 and 6.2 in Evans. Do not worry too much about all the details in the proofs, we will worry about those in the lectures. However, try to form a general picture of the material covered.

5. Write the Dirichlet problem

$$\Delta u = f \in L^2(U),\ u|_{\partial U} = 0$$

   explicitly in the weak form.

6. Why the Lax-Milgram lemma is not a direct corrollary of the Riesz representation theorem?