Partial Differential Equations II

HW 2

Return to Jussi by Monday, September 29.

1. Assume that U and V are open subsets of $\mathbb{R}^n$ with $\overline{V} \subset U$. Show that there exists a $u \in C_0^\infty(U)$ such that u is identically $= 1$ in V.

2. Assume that $U \subset \mathbb{R}^d$ is bounded and open, and $\overline{U} \subset \cup_{i=1}^N V_i$, where V_i are open. Show, that there exists $\zeta_i \in C^\infty(V_i)$ such that $0 \leq \zeta_i \leq 1$, $\operatorname{supp} \zeta_i \subset V_i$ and that

$$\sum_{i=1}^N \zeta_i(x) = 1 \text{ for all } x \in U.$$

Before the next exercise recall the definition and basic properties of absolutely continuous functions as explained in the Real Analysis course, or for example in Rudin's *Real and Complex Analysis*.

3. Take $n = 1$ and assume that $u \in W^{1,p}(0,1)$, $1 \leq p < \infty$. Show that u is a.e equal to an absolutely continuous function, and it thus has a pointwise derivative u' a.e. and that $u' \in L^p(0,1)$. **Hint**: Remember, that every sequence converging in L^p has a subsequence converging pointwise a.e. .

4. Prove that if $u \in W^{1,p}(0,1)$, $1 < p < \infty$, then

$$|u(x) - u(y)| \leq |x-y|^{1-1/p} \left(\int_0^1 |u'|^p \, dt \right)^{1/p} \text{ for a.a } x, \, y \in (0,1).$$

Before trying the next questions, read sections 5.6 and 5.7 in Evans. Do not worry too much about all the details in the proofs, we will worry about those in the lectures. However, try to form a general picture of the material covered.

5. Explain what is meant by a compact linear map between Banach spaces.

6. Assume U is bounded. What can you say about the compactness of the embedding $W^{1,p}(U) \to L^p(U)$?