II TOISEN KERTALUVUN ELLIPTISET YHTÄLÖT

2.0 <u>Johdatteleva esimerkki</u>

Olkoon $U \subset \mathbb{R}^d$, $d \geq 2$, rajj. C^1-alue. Tarkastellaan ns. reuna-arvo-ongelmaa Laplace-yhtälölle:

$$(D) \quad \begin{cases} \Delta u = f, & f \in L^2(U). \\ u|_{\partial U} = 0 \end{cases}$$

Haluamme löytää siis funktion u s.e. u:lla on toisen kl:n heikot derivaatat ja

$$\Delta u := \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_d^2} = f,$$

ja edelleen trace-mielessä $u|_{\partial U} = 0$. Tuntuisi että luonnollisella siis vaatia $u \in H_0^2(U)$.

Toimimme kuitenkin toisin: olk. $\phi \in C_0^1(U)$. Tällöin, jos $u \in C^2(U)$, $\Delta u = f$, niin

$$\int_U \phi \bar{f} \, dx = \int_U \phi \overline{\Delta u} \, dx = -\int_U \langle \nabla\phi, \nabla\bar{u} \rangle \, dx$$

$\uparrow$ $\phi, \partial_\nu \phi = 0$ ∂U:lla

Olkoon nyt

$$B_U(\phi, \psi) := \int_U \langle \nabla\phi, \nabla\bar{\psi} \rangle, \quad \phi, \psi \in H^1(U)$$

B_U on ns. Dirichlet-muoto U:ssa.

Jos $u \in C^2(U)$, niin

$$\begin{cases} \Delta u = f & U\text{:ssa} \\ u|_{\partial U} = 0 \end{cases} \Rightarrow \begin{cases} B_U(\phi, u) = -\lambda_f(\phi), \\ \lambda_f(\phi) := \int_U \phi \bar{f} \, dx \end{cases} \quad \forall \phi \in H_0^1(U).$$

<u>Määr.</u> Sanomme, että $u \in H_0^1(U)$ on (D):n <u>heikko</u> <u>ratkaisu</u>, jos $B_U(\phi, u) = \lambda_f(\phi) \quad \forall \phi \in H_0^1(U)$.

Nyt huomaamme / muistamme seuraavat asiat:

A) $H_0^1(U)$ on Hilbert-avaruuden $H^1(U)$ suljettu aliavaruus, ja siis itsekin Hilbert-avaruus sisätulon

$$(u, v)_{H^1} := \int_U \langle \nabla u, \nabla \bar{v} \rangle + u\bar{v} \, dx$$

suhteen.

B) $H_0^1(U)$:ssa H^1-normi on ekvivalentti normin

$$\|u\|^2_{H_0^1(U)} = \int_U |\nabla u|^2 dx$$

kanssa.

C) A), B) $\Rightarrow$ $H_0^1(U)$ on Hilbert-avaruus myös sisätulon

$$\langle u,v\rangle_{H_0^1} := \int_U \langle \nabla u, \nabla \bar{v}\rangle \, dx = B_U(u,v)$$

suhteen !

D) $H_0^1(U) \ni \phi \longmapsto -\int \bar{f}\phi \, dx = -\lambda_f(\phi)$ on $H_0^1(U)$:n lineaarinen funktionaali : lineaarisuus on selvä, ja

$$|-\lambda_f(\phi)| \leq \|f\|_{L^2(U)} \|\phi\|_{L^2(U)} \leq C_U \|f\|_{L^2(U)} \|\phi\|_{H_0^1(U)}$$

E) Rieszin esityslauseen nojalla on olemassa yksikäsitteinen $u \in H_0^1(U)$ s.e.

$$-\lambda_f(\phi) = \langle \phi, u\rangle = B_U(\phi, u),$$

ja olemme osoittaneet, että (D):llä on yksikäsitteinen heikko ratkaisu !

Haluamme yleistää tämän yleisimmille toisen kl:n ongelmille.

2.1. Elliptic equations (Sorry about the Finnish above ! (☺))

Let's consider an operator (in $\mathbb{R}^n$, $n=1,2,\dots$)

$$Lu = -\sum_{i,j=1}^n a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_i} + c(x) u .$$

Here a^{ij}, b^i and c are (at least) measurable coefficients.

Sometimes one also consider ops given in divergence

form

$$Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a^{ij}(x) \frac{\partial}{\partial x_j} u\right) + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_i} + c(x) u .$$

We want to solve

$$\begin{cases} Lu = f & \text{in } U \\ u|_{\partial U} = 0 \end{cases} \qquad f \text{ given } U \to \mathbb{C}$$

$\leftarrow$ in some sense - usually trace !

We make two assumptions on a^{ij} :

A) (Symmetry) $a^{ij} = a^{ji}$ and $a^{ij} \in \mathbb{R}$ $\forall i,j$

B) (Uniform) Ellipticity : $\exists$ const $\theta > 0$ s.t.

$$\sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2$$

i.e. the quadratic form determined by the real symmetric matrix (a^{ij}) is positive definite.

Examples : i) Δ corresponds to $b^i \equiv c \equiv 0$,

$$a^{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}.$$

This is elliptic. $(x' = (x_2, \dots, x_n))$

ii) $\frac{\partial^2}{\partial x_1^2} - \Delta_{x'}$ is not elliptic (wave op.)

iii) $\frac{\partial}{\partial x_1} - \Delta_{x'}$ (Heat op)

$i\frac{\partial}{\partial x_1} - \Delta_{x'}$ (Schrödinger op) are also not elliptic.

2.2. Weak solutions

Since a^{ij} are real valued, and we also assume b^i & c to be real valued, we may just as well restrict ourselves – for the moment – to real valued f and u. Hence we consider

$$H^1_0(U) = \{ u \in L^2(U) ;\ u, Du \in L^2(U),\ Tu = 0 \}$$

with norm

$$\|u\|^2_{H^1_0(U)} = \int_U |\nabla u|^2 \, dx$$

and inner product

$$\langle u, v \rangle_{H^1_0(U)} = \int_U \langle \nabla u, \nabla v \rangle \, dx$$

← Notice that this is bilinear, not just conjugate linear.

We assume now

$$a^{ij},\ b^i,\ c \in L^\infty(U)$$

Assume also that L is in the divergence form. Assuming for the moment $u \in C^2(U)$, $v \in C^1_0(U)$ we get

$$\int_U f v \, dx = \int_U - \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a^{ij} \frac{\partial u}{\partial x_j}\right) v \, dx + \int_U b^i(x) \frac{\partial u}{\partial x_i} v + \int_U c(x) u v \, dx$$

$$\overset{\text{ind}}{\underset{\text{as}}{=}} \int_U \sum_{i,j=1}^n a^{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_j} + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_i} v + c(x) u v \, dx .$$

Note that if $a^{ij} \in L^\infty(U)$ it is not obvious how to interpret $\frac{\partial}{\partial x_i}\left(a^{ij} \frac{\partial u}{\partial x_j}\right)$, however the bilinear form corresponding to L,

$$B[u,v] := \int_U \sum_{i,j=1}^n a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x_i} v + c(x) u v \, dx ,$$

$$u, v \in H^1_0(U) ,$$

is well defined.

<u>Def.</u> $u \in H^1_0(U)$ is a weak solution of b.v.p

$$\begin{cases} Lu = f & \text{in } U \\ u|_{\partial U} = 0 \end{cases} , \quad L = \sum_{i,j} \frac{\partial}{\partial x_j} a^{ij} \frac{\partial}{\partial x_j} + \sum_i b^i \frac{\partial}{\partial x_i} + c(x)$$

if

$$B[u,v] = (f,v) \quad \left(= \int_U f v \, dx \right)$$

$\forall\ v \in H^1_0(U)$.

Note that to prove existence, we cannot resort to Riesz rep thm since $B[\cdot,\cdot]$ may not be symmetric i.e. not an inner product.

2.3. Lax-Milgram Thm.

This generalizes Riesz rep. thm to case of non-symmetric B.

Let H be a real Hilbert space, with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let $\langle \cdot,\cdot \rangle$ be the canonical pairing

$$(x,\lambda) \longmapsto \lambda(x), \quad x \in H, \ \lambda \in H' = \text{dual of } H$$
$$(\approx H \text{ by Riesz})$$

<u>Thm</u> (Lax-Milgram) Assume

$$B : H \times H \longrightarrow \mathbb{R}$$

is a bilinear mapping s.t. $\exists$ constants $\alpha, \beta > 0$ s.t.

(i) $\quad |B[u,v]| \leq \alpha \|u\| \|v\| \quad$ (boundedness)

(ii) $\quad \beta \|u\|^2 \leq B[u,u] \quad$ (coercivity).

Let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then $\exists$ unique $u \in H$ s.t.

$$B[u,v] = f(v) \quad \forall v \in H.$$

Pf. <u>Step I</u> For a fixed $u \in H$, mapping

$$H \ni v \longmapsto B[u,v]$$

is a bounded linear functional on H. Hence Riesz $\Rightarrow$ $\exists$ unique $w \in H$ s.t.

(*) $\quad B[u,v] = (w,v), \quad v \in H.$

Define $Au := w$ if (*) holds. This is a well defined map $H \to H$.

<u>Step 2</u> ($A \in \mathcal{L}(H)$). Note that A is linear: $\lambda_1, \lambda_2 \in \mathbb{R}$, $u_1, u_2 \in H$, then

$$(A(\lambda_1 u_1 + \lambda_2 u_2), v) = B[\lambda_1 u_1 + \lambda_2 u_2, v]$$
$$= \lambda_1 B[u_1,v] + \lambda_2 B[u_2,v]$$
$$= \lambda_1 (Au_1, v) + \lambda_2 (Au_2, v)$$
$$= (\lambda_1 Au_1 + \lambda_2 Au_2, v) \quad \forall v \in H$$

$\Rightarrow A(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 Au_1 + \lambda_2 Au_2.$

$\therefore A$ linear. Also,

$$\|Au\|^2 = (Au, Au) = B[u, Au] \leq \alpha \|u\| \|Au\|,$$

Hence

$$\|Au\| \leq \alpha \|u\|$$

i.e. A is bounded.

<u>Step 3</u> Note that

$$\beta \|u\|^2 \leq B[u,u] = (Au,u) \leq \|Au\| \|u\|,$$

Hence

$$\|Au\| \geq \beta \|u\|.$$

i.e. A is bounded from below. Thus A is one-to-one. Also, if $v_i = Au_i$, and $v_i \longrightarrow v$, then

$$\|v_i - v_j\| = \|A(u_i - u_j)\| \geq \beta \|u_i - u_j\|,$$

and hence (u_i) is a Cauchy sequence in H i.e. $\exists$ limit

$$u := \lim u_i .$$

Then A cont $\Rightarrow$

$$Au = \lim Au_i = \lim v_i = v$$

i.e. $Im(A) = \{v \in H;\ v = Au \text{ for some } u\}$ is a closed subspace of H.

<u>Step 4</u> Next we prove that A is onto i.e. $Im(A) = H$.

Assume not. Then since $Im(A)$ is a closed subspace, $\exists\ w \in Im(A)^{\perp}$, $w \neq 0$. But then

$$0 = (\underset{\in Im(A)}{Aw}, \underset{\in Im(A)^{\perp}}{w}) = B[w,w] \geq \beta \|w\|^2 \Rightarrow w = 0 \ \lightning .$$

<u>Step 5</u> (Existence) By Riesz rep thm $\exists$ unique $w \in H$ s.t.

$$f(v) = (w, v) \quad \forall\, v \in H .$$

Then A onto $\Rightarrow \exists\, u \in H$ s.t. $Au = w$. Hence

$$B[u,v] = (Au, v) = (w, v) = f(v) \quad \forall\, v \in H,$$

and we have shown existence.

<u>Step 6</u> (Uniqueness) Assume $\exists\ u, \tilde{u} \in H$ s.t.

$$B[u,v] = f(v) = B[\tilde{u}, v] \quad \forall\, v \in H .$$

Hence

$$0 = B[u - \tilde{u}, v] \quad \forall\, v \in H .$$

Choose $v = u - \tilde{u}$. Then

$$0 = B[u - \tilde{u}, u - \tilde{u}] \geq \beta \|u - \tilde{u}\|^2$$

$\Rightarrow u = \tilde{u}$. $\square$

2.4. Energy estimates

We want to apply Lax-Milgram to the bilinear form def by

$$L = -\sum_{i,j} \frac{\partial}{\partial x_i} a^{ij} \frac{\partial u}{\partial x_j} + \sum_i b^i(x) \frac{\partial u}{\partial x_i} + cu ,$$

i.e. to

$$B[u,v] = \int_U \sum_{i,j=1}^{n} a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_i b^i(x) \frac{\partial u}{\partial x_i} v + cuv \, dx ,$$

To do this we will need estimates:

<u>Thm 1</u> i) (Boundedness) $\exists$ const $\alpha > 0$ s.t.

$$|B[u,v]| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} \quad \forall\, u, v \in H_0^1(U)$$

ii) ("Gårdings inequality") $\exists$ constants $\beta > 0$ and $\gamma \geq 0$ s.t.

$$\beta \|u\|^2_{H_0^1(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)} \quad \forall\, u \in H_0^1(U) .$$

Pf. <u>Step 1</u> Let $u, v \in H_0^1(U)$. Then C-S

$$|B[u,v]| \leq \sum_{i,j} \|a^{ij}\|_{L^\infty(U)} \int_U |\nabla u||\nabla v|\,dx$$

$$+ \sum_i \|b^i\|_{L^\infty(U)} \int_U |\nabla u||v|\,dx + \|c\|_{L^\infty(U)} \int_U |u||v|\,dx$$

$$\overset{\text{Poincaré}}{\leq} \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$$

Proving (i).

<u>Step 2</u> By ellipticity:

$$\theta \int_U |\nabla u|^2 dx \leq \int_U \sum_{i,j} a^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\,dx$$

$$= B[u,u] - \int_U \sum_i b^i \frac{\partial u}{\partial x_i} u + cu^2\,dx$$

$$\leq B[u,u] + \sum_i \|b^i\|_{L^\infty(U)} \underbrace{\int_U |\nabla u||u|\,dx}_{\text{trouble maker!}} + \|c\|_{L^\infty(U)} \|u\|^2_{L^2(U)}$$

Now gen. Cauchy

$$\int_U |\nabla u||u|\,dx \leq \varepsilon \int_U |\nabla u|^2 dx + \frac{1}{4\varepsilon} \int_U u^2\,dx \quad \forall \varepsilon > 0$$

Hence

$$\theta \int_U |\nabla u|^2 dx - \varepsilon \Big(\sum_i \|b^i\|_{L^\infty(U)}\Big) \int_U |\nabla u|^2 dx \leq$$

$$\leq B[u,u] + \left\{ \frac{\sum_i \|b^i\|_{L^\infty(U)}}{4\varepsilon} + \|c\|_{L^\infty(U)} \right\} \int_U u^2\,dx$$

Choose now $\varepsilon > 0$ so small that

$$\theta - \varepsilon \Big(\sum_i \|b^i\|_{L^\infty(U)} \Big) < \frac{\theta}{2}.$$

Then the claim follows. ▯

This is <u>not</u> yet enough to give weak solvability for

$$Lu = f, \; u \in H_0^1(U)$$

via Lax-Milgram.

However we can prove the following:

<u>Th.</u> 2 $\exists$ const $\gamma \geq 0$ s.t. $\forall \mu \geq \gamma$ and $f \in L^2(U)$ $\exists$ unique weak $H_0^1(U)$ sol of

$$\begin{cases} Lu + \mu u = f & \text{in } U \\ u|_{\partial U} = 0. \end{cases}$$

Pf. Let γ be as in our Thm 2; then the bilinear form corresponding to $Lu + \mu u$ is

$$B_\mu[u,v] := B[u,v] + \mu \int_U uv\,dx$$

Hence if $\mu \geq \gamma$, cond. of Lax-Milgram are valid for $B_\mu[u,v]$.

Given $f \in L^2(U)$, let

$$\langle f, v \rangle = \int_U f v \, dx ;$$

Then this is a bnd linear functional on $H^1_0(U)$ and Lax-Milgram $\Rightarrow$ $\exists$ unique $u \in H^1_0(U)$ s.t.

$$B_\mu[u,v] = \langle f, v \rangle \quad \square$$

2.5. Fredholm's Alternative I

In order to be able to say something directly about the (weak) solvability of

$$\begin{cases} Lu = f \in L^2(U) \\ u|_{\partial U} = 0 \end{cases}$$

we need to study also $\mu < \gamma$. For this we use Fredholm's alternative:

Thm. 2.5.1. Assume H Hilbert, $K : H \to H$ cpt. Then

(i) $\ker(I-K)$ is finite dimensional

(ii) $\operatorname{im}(I-K)$ is closed

(iii) $\operatorname{im}(I-K) = \ker(I-K^*)^\perp$ ($\Rightarrow$ $\operatorname{im}(I-K)$ has finite codimension)

(iv) $\ker(I-K) = \{0\} \Leftrightarrow \operatorname{im}(I-K) = H$

∇ (i.e. $I-K$ injective $\Leftrightarrow$ $I-K$ surjective) ∇

(v) $\dim(\ker(I-K)) = \dim \ker(I-K^*)$

Pf. (i) Assume $\dim \ker(I-K) = \infty$.

Then $\exists$ inf. ON set $u_n \in \ker(I-K)$ i.e.

$$\begin{cases} \|u_n\| = 1, \; n = 1, 2, \ldots \\ u_n \perp u_m, \quad n \neq m \\ (I-K)u_n = 0 . \end{cases}$$

But then

$$K u_n = u_n$$

$\overset{K}{\underset{\text{cpt}}{\Rightarrow}}$ $\exists$ subsequence (u_{n_j}) s.t. $\exists \lim_j K u_{n_j} = v$.

Hence $v = \lim u_{n_j}$, but this is not possible since

$$i \neq j \Rightarrow \|u_{n_i} - u_{n_j}\| = \sqrt{2} .$$

$\therefore \dim \ker(I-K) < \infty$.

(ii) We want to prove a lower bound for $I-K$;

Since $\ker(I-K)$ might not be trivial, the best we can hope is

$$(LB_\gamma) \begin{cases} \|u - Ku\| \geq \gamma \|u\| \quad \forall \; u \in \ker(I-K)^\perp \\ \text{for some } \gamma > 0 \end{cases}$$

Assume not: Then $\forall k \in \{1,2,\dots\}$ $\exists$ $u_k \in \ker(I-K)^\perp$ s.t.

$$\|u_k\| = 1, \quad \|u_k - Ku_k\| < \frac{1}{k}.$$

Hence

(i) $$\lim \|u_k - Ku_k\| = 0$$

$\{u_k\}$ bnd $\Rightarrow$ $\exists$ weakly convergent subseq. (u_{k_j}) i.e. (B weak cpt)

$\exists$ $u \in H$ s.t.

$$(u_{k_j}, v) \longrightarrow (u, v) \quad \forall v \in H.$$

K cpt $\Rightarrow$

$$Ku_{k_j} \longrightarrow Ku \quad \text{in } H.$$

(i) $\Rightarrow$ $$u_{k_j} \longrightarrow \underset{Ku}{\overset{\|}{u}} \quad \text{in } H.$$

$\therefore$ $u \in \ker(I-K)$ and thus

$$(\underset{\ker(I-K)^\perp}{\overset{\ni}{u_{k_j}}}, u) = 0 \quad \forall k_j$$

But then $(\lim \underset{1}{\|u_{k_j}\|} = \|u\|)$

$$1 = (u,u) = \lim_j (u_{k_j}, u) = 0 \quad ⚡.$$

Hence $(LB)_\gamma$ holds for some $\gamma > 0$.

Assume now

$$v = \lim_j (u_j - Ku_j), \quad u_j \in \ker(I-K)^\perp$$ ← think why so!

Let $v_j = u_j - Ku_j$. Then

$$\|v_i - v_j\| \geq \gamma \|u_i - u_j\|$$

$\Rightarrow$ (u_j)'s Cauchy $\Rightarrow$ $\exists \lim u_j = u \in H$. Then

$$u - Ku = \lim (u_j - Ku_j) = v,$$

and $\operatorname{im}(I-K)$ is closed.

(iii) We always have $(A \in \mathcal{L}(H))$

$$\overline{\operatorname{im} A} = (\ker A^*)^\perp :$$

Let $y \in \operatorname{im}(A)$, $y = Ax$, $x \in H$. Then $\forall$ $z \in \ker A^*$:

$$(y,z)_H = (Ax, z)_H = (x, A^*z) = 0$$

i.e. $y \in (\ker A^*)^\perp$ $\Rightarrow$ $\operatorname{im}(A) \subset (\ker A^*)^\perp$ $\Rightarrow$ $\overline{\operatorname{im}(A)} \subset (\ker A^*)^\perp$.
↑ closed

Conv. Assume $z \in \operatorname{im}(A)^\perp$. Then $\forall x \in H$

$$0 = (z, Ax) = (A^*z, x)$$

$\Rightarrow$ $\operatorname{im}(A)^\perp \subset \ker A^*$ $\Rightarrow$ $\overline{\operatorname{im}(A)} \supset \ker(A^*)^\perp$. $\square$

Hence $\operatorname{im}(I-K)$ closed $\Rightarrow$

$$\operatorname{im}(I-K) = \ker(I-K^*)^\perp.$$

(iv) Assume $\ker(I-K)=\{0\}$.

Let's assume $\underset{\substack{\| \\ H_1}}{\operatorname{im}(I-K)} \subsetneq H$.

H_1 closed. Let $H_2=(I-K)H_1$. Then $H_2 \subsetneq H_1$.

if not then $\forall z_1 \in H$ $\exists$ $y_1 \in H_2$ s.t.

$$(I-K)y_1 = x_1 = (I-K)z_1 \quad \text{since } x_1 \in H_1$$

$\therefore$ $I-K$ inj $\Rightarrow y_1 = z_1 \Rightarrow \underset{\substack{\| \\ H}}{H_2} \subset H_1$ ↯.

Hence

$$H_2 \subsetneq H_1 \subsetneq H$$

Similarly (Ind.) if $H_k=(I-K)H_{k-1}=\dots=(I-K)^k H$, we have

$$\begin{cases} \subsetneq H_{k+1} \subsetneq H_k \subsetneq \dots \subsetneq H_1 \subsetneq H. \\ H_{k+1} \text{ a closed subspace } H \end{cases}$$

Choose $u_k \in H_k$, $\|u_k\|=1$, $u_k \in H_{k+1}^{\perp}$. Then

$$Ku_k - Ku_\ell = -(\mathbb{I}_k - Ku_k) + (\mathbb{I}_\ell - Ku_\ell) + u_k - \underset{\substack{\cap \\ H_{\ell+1}^{\perp}}}{u_\ell}.$$

Hence, $k>l \Rightarrow$

$$\begin{cases} H_{k+1} \subsetneq H_k \subseteq H_{\ell+1} \subsetneq H_\ell \\ u_k - Ku_k \in H_{k+1} \subsetneq H_{\ell+1} \qquad \Rightarrow \|Ku_k - Ku_\ell\| \geq \|u_\ell\| = 1 \\ u_\ell - ku_\ell \in H_{\ell+1} \\ u_k \in H_k \subset H_{\ell+1} \end{cases}$$

This is a contradiction since K cpt.

$\therefore$ $\operatorname{im}(I-K)=H$.

Conversely, if $\operatorname{im}(I-K)=H \Rightarrow \ker(I-K^*)=\{0\}$
$\overset{K^* \text{ cpt}}{\Rightarrow}$ $\operatorname{im}(I-K^*)=H \Leftrightarrow \ker(I-K)=\{0\}$.

(v) Next we prove

(ii) $\dim \ker(I-K) \geq \dim \operatorname{im}(I-K)^{\perp}$.

Assume not. Then $\exists$ bnd. linear op

$$\begin{cases} A: \ker(\mathbb{I}-K) \longrightarrow \operatorname{im}(I-K)^{\perp} \\ A \text{ inj, but \underline{not} surj.} \end{cases}$$

<u>Note</u> $\begin{cases} \ker(\mathbb{I}-K),\ \operatorname{im}(\mathbb{I}-K)^{\perp} \\ \text{are fin. dimensional} \end{cases}$

Extend $A: H \longrightarrow \operatorname{im}(I-K)^{\perp}$

by $A|_{\ker(\mathbb{I}-k)^{\perp}} = 0$.

A fin. rank $\Rightarrow A$ cpt $\Rightarrow K+A$ is cpt.

Then $\ker(\mathbb{I}-(K+A))=\{0\}$.

Pf of this:

$$u-Ku-Au=0 \Leftrightarrow u-Ku=Au \in \operatorname{im}(\mathbb{I}-K)^{\perp}$$

$$\Rightarrow u-Ku=Au=0 \Rightarrow \begin{cases} u \in \ker(\mathbb{I}-K) \\ Au=0 \end{cases} \Rightarrow u=0.$$

Apply (iv) to $K+A$. Then

$$\mathrm{im}(\mathbb{I}-(K+A)) = H.$$

This will not do: choose $v \in \mathrm{im}(\mathbb{I}-K)^\perp$, $v \notin \mathrm{im}(A)$. Then

$$u-(Ku+Au) = v$$

has no solution for

$$\underbrace{-Au}_{\in\, \mathrm{im}(\mathbb{I}-K)^\perp} = \underbrace{v}_{\in\, \mathrm{im}(\mathbb{I}-K)^\perp} + \underbrace{(u-Ku)}_{\in\, \mathrm{im}(\mathbb{I}-K)}.$$

Now

$$\mathrm{im}(\mathbb{I}-K^*)^\perp = \ker(\mathbb{I}-K)$$

(i) $\Rightarrow$

$$\dim\ker(\mathbb{I}-K^*) \geq \dim \mathrm{im}(\mathbb{I}-K^*)^\perp = \dim\ker(\mathbb{I}-K)$$

and by replacing K^* with K we also get

$$\dim\ker(\mathbb{I}-K) \geq \dim\ker(\mathbb{I}-K^*),$$

Proving (v). $\square$

2.6. Fredholm's alternative II

Def. (i) Let $Lu = -\nabla\cdot A\nabla u + \langle B, \nabla u\rangle + cu$. The formal adjoint L^* of L is

$$L^*v = -\nabla\cdot A^t\nabla v - \nabla\cdot(Bu) + cu$$

$$= \sum_{i,j} \partial_{x_i} a^{ij}(x)\partial_{x_j} v - \Big(\sum_{i=1}^n \partial_{x_i} b^i\Big)u - \sum_{i=1}^n b^i(x)\partial_{x_i} u + cu$$

$$= \sum_{i,j} \partial_{x_i} a^{ij}(x)\partial_{x_j} v - \sum_{i=1}^n b^i(x)\partial_{x_i} u + (c-\nabla\cdot B)u$$

[for this to make sense we need $\nabla\cdot B \in L^\infty$!

(ii) If B is the bilinear form corresponding to L, then the adjoint bilinear form B^* is

$$B^*: H^1_0(U)\times H^1_0(U) \to \mathbb{R},$$
$$B^*[v,u] = B[u,v]$$

Note: B^* corresponds formally to L^*.

(iii) We define, that $v \in H^1_0(U)$ is a weak sol of the adjoint problem

$$\begin{cases} L^*v = f & \text{in } U \\ v = 0 & \text{on } \partial U\end{cases} \qquad (f \in L^2(U))$$

if

$$B^*[v,u] = \int f u\,dx \quad \forall u \in H^1_0(U).$$

Q: Is always $B_{L^*} = B_L^*$?

Thm (Fredholm's alternative for L) Either

(i) The b.v.p

$$(\star)\quad \begin{cases} Lu = f \in L^2(U) \\ u|_{\partial U} = 0\end{cases}$$

has a unique weak solution.

$$(\star)_0 \quad \begin{cases} Lu = 0 \text{ in } U \\ u|_{\partial U} = 0\end{cases}$$

OR

(ii) $\exists\, u \neq 0$, $u \in H^1_0(U)$, solving $(\star)_0$. Then the dimension d of the subspace

$$N = \{u \in H^1_0(U);\ u \text{ a weak sol of } (\star)_0\}$$

is >0 and finite, and also

$$\dim\{v \in H^1_0(U);\ v \text{ a weak sol of } L^*v = 0,\ v|_{\partial U} = 0\} = d.$$

(iii) Finally, (☺) has a weak solution iff

$$(f, v) = 0 \quad \forall\, v \in N^*.$$

Pf. Step I: Choose γ s.t. Lax-Milgram holds for

$$B_\gamma[u,v] := B[u,v] + \gamma(u,v)$$

B_γ corresponds to $L_\gamma u := Lu + \gamma u$,

Then $\forall\, g \in L^2(U)$ $\exists$ <u>unique</u> $u \in H^1_0(U)$ s.t.

(☺) $\quad B_\gamma[u,v] = (g, v) \quad \forall\, v \in H^1_0(U).$

Def

$$L_\gamma^{-1} g := u \quad \text{if} \quad (☺) \text{ holds}.$$

Lax-Milgram (uniqueness etc...) $\Rightarrow L_\gamma^{-1} : L^2(U) \to H^1_0(U)$ is linear.

<u>Step II</u>: Since B_γ strictly coercive, $(u = L_\gamma^{-1} g)$ $\quad (\beta > 0)$

$$\beta \| L_\gamma^{-1} g\|^2_{H^1_0} \le B_\gamma[u,u] = (g,u) \le \|g\|_{L^2} \|L_\gamma^{-1} g\|_{L^2}$$

$$\therefore \quad \|L_\gamma^{-1} g\|_{H^1_0(U)} \le C \|g\|_{L^2(U)}$$

i.e. $L_\gamma^{-1} : L^2(U) \to H^1_0(U)$ is bounded.

<u>Step III</u>: Now if $u \in H^1_0(U)$,

$$B[u,v] = (f, v) \quad \forall\, v \in H^1_0(U)$$

$$\Leftrightarrow \quad B_\gamma[u,v] = (\gamma u + f, v) \quad \forall\, v \in H^1_0(U)$$

$$\Leftrightarrow \quad u = L_\gamma^{-1}(\gamma u + f). \quad (☺)$$

Define

$$Ku := \gamma L_\gamma^{-1} u.$$

Then (☺) $\Leftrightarrow$

$$(\mathbb{I} - K)u = h, \quad h = L_\gamma^{-1} f \in H^1_0(U).$$

Now

$$K : L^2(U) \to H^1_0(U) \subset\subset L^2(U)$$

and hence we can apply Fredholm's alternative th. $(\mathbb{I} - K)$ on $L^2(U)$.

<u>Step IV</u>: Note now that

$$u - Ku = L_\gamma^{-1} f, \quad u \in L^2(U)$$

$$\Leftrightarrow \begin{cases} B[u,v] = (f,v) \ \forall\, v \in H^1_0(U). \\ u \in H^1_0(U) \end{cases}$$

i.e. $u - Ku = L_\gamma^{-1} f$, $u \in L^2(U)$

$\Leftrightarrow$

u a weak sol of $\begin{cases} Lu = f \\ u|_{\partial U} = 0. \end{cases}$

Assume now that $(\mathbb{I} - K)u = 0 \Rightarrow u = 0$.

Then above Fredholm $\Rightarrow$ $\begin{cases} Lu = f \in L^2(U) \\ u|_{\partial U} = 0 \end{cases}$ has always a unique weak sol.

$(\mathbb{I} - K^*)v = 0$

$\Leftrightarrow \quad v \perp \operatorname{im}(\mathbb{I} - K)$

i.e. $\forall \ g \in L^2$,

$$(g - \gamma L_\gamma^{-1} g, v) = 0$$

$\Leftrightarrow$ $\qquad u \in L_\gamma^{-1} g$

$$\underset{\| }{(g,v) - \gamma(\underbrace{L_\gamma^{-1} g}_{=u}, v)} \quad \forall u \in H_0^1(U)$$

$B_\gamma[u, v]$

$\Leftrightarrow \ B[u,v] = 0 \quad \forall \ u \in H_0^1(U)$

$\Leftrightarrow \ B^*[v,u] = 0 \quad \forall \ u \in H_0^1(U)$.

$\therefore \quad \ker(\mathbb{I} - K^*) = \{ v \in H_0^1(U);\ L^* v = 0 \text{ weakly} \}$ $\square$

2.7. Interior regn

Consider eqn

$$-\Delta u = f \quad , \quad f \in L^2(\mathbb{R}^n).$$

Assume $u \in H^1(\mathbb{R}^n)$ is a (distributive) solution

i.e.

$$\int \langle \nabla u, \nabla \phi \rangle = \int u(-\Delta \phi)\,dx = \int f \phi\,dx \ \forall \ \phi \in C_0^\infty$$

Then

$$-\Delta u + u = f + u \in L^2(\mathbb{R}^n)$$

i.e. by taking Fourier-transform:

$$(|\xi|^2 + 1)\hat{u} = \hat{f} + \hat{u} =: \hat{g} \in L^2(\mathbb{R}^n)$$

$$\Rightarrow \quad \hat{u}(\xi) = \frac{\hat{g}(\xi)}{|\xi|^2 + 1}, \quad \hat{g} \in L^2(\mathbb{R}^n).$$

Hence

$$\int (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \simeq \int (1+|\xi|^2)^{s-2} |\hat{g}(\xi)|^2 d\xi < \infty$$

if $s \leq 2$.

$$\therefore \quad \begin{cases} -\Delta u = f \in L^2(\mathbb{R}^n) \\ u \in H^1(\mathbb{R}^n) \end{cases} \Rightarrow \quad u \in H^2(\mathbb{R}^n).$$

We can do a ~~similar~~ analogous arguments for bilinear forms:

Assume again

$$-\Delta u = f \in L^2(\mathbb{R}^n) \quad \& \quad f \text{ decays "fast enough".}$$

Then

$$\int_{\mathbb{R}^n} f^2\,dx = \int_{\mathbb{R}^n} (\Delta u)^2\,dx = \sum_{i,j} \int \frac{\partial^2 u}{\partial x_i^2} \frac{\partial^2 u}{\partial x_j^2}\,dx$$

$$= -\sum_{i,j}\int \frac{\partial^3 u}{\partial x_j \partial x_i^2}\frac{\partial u}{\partial x_j}\,dx = \sum_{i,j}\int \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\,dx = \int |D^2 u|\,dx$$

$\therefore$ one can bound the H^2 norm in terms of $\|f\|_{L^2}$ provided all integrations by part above are valid (this is not always trivial – or even justifiable!)

We want to prove similar kinds of results for H^1-sols of

$$\begin{cases} Lu = f \in L^2(U) \\ u|_{\partial U} = 0 \end{cases} \qquad u \in H^1(U)$$

in $V \subset\subset U$. Here

$$Lu = -\nabla\cdot A(x)\nabla u + \langle B(x), \nabla u\rangle + c(x)u.$$

<u>Thm. 1</u> Assume

$A = (a^{ij})$, $a^{ij} \in C^1(U)$,

$b^i, c \in L^\infty(U)$.

If $u \in H^1(U)$ solves $Lu = f \in L^2(U)$ weakly in U

i.e. $B[u,v] = \int_U fv \quad \forall\, v \in H^1_0(U)$

Then

(i) $u \in H^2_{loc}(U)$ $\quad \left(H^2_{loc}(U) = \{ v \in L^1_{loc}(\mathbb{R});\ \phi v \in H^2(U)\ \forall \phi \in C_0^\infty(U)\} \right)$

and $\forall\, V \subset\subset U$ we have

$$\|u\|_{H^2(V)} \le C_{U,V}\left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

Pf. <u>Step I</u> Fix W s.t.

$$V \subset\subset W \subset\subset U$$

and then choose $\zeta \in C_0^\infty(U)$ s.t.

$$\begin{cases} \zeta \equiv 1 \text{ on } V,\ \zeta \equiv 0 \text{ in } U \setminus W \\ 0 \le \zeta \le 1 \end{cases}$$

<u>Step II</u> $B[u,v] = \int fv \quad \forall\, v \in H^1_0(U)$

$$\Rightarrow \sum_{i,j}\int_U a^{ij} u_{x_i} v_{x_j}\,dx = \int_U \tilde{f} v, \quad \tilde{f} = f - \sum_i b^i(x)\partial_{x_i}u - cu \in L^2(U).$$

<u>Step III</u> Define

$$D^h_k u(x) = \frac{u(x+he_k) - u(x)}{h},$$

$k \in \{1,\dots,n\}$, $x \in V$, $|h|$ small enough, $h \neq 0$.

Choose

$$v = -D^{-h}_k(\zeta^2 D^h_k u) \qquad \text{"}2^{nd}\text{ diff. quotient"}$$

$\in H^1_0(U)$ when $|h|$ small enough.

Let $A = \int_U \langle A(x)\nabla u, \nabla v\rangle\,dx$, $B = \int_U \tilde{f} u\,dx$.

Step IV

$$A = -\sum_{i,j}\int_U a^{ij} u_{x_i}\left[D_k^{-h}(\zeta^2 D_k^h u)\right]_{x_j} dx$$

$$= \sum_{i,j}\int_U D_k^h(a^{ij}u_{x_i})(\zeta^2 D_k^h u)_{x_j}\, dx$$

$$= \sum_{i,j}\int_U a^{ij,h}(D_k^h u_{x_i})(\zeta^2 D_k^h u)_{x_j}\, dx$$

$$+ \sum_{i,j}\int_U (D_k^h a^{ij}) u_{x_i}(\zeta^2 D_k^h u)_{x_j}\, dx$$

$\int f D_k^h g$

$= \int f \, \frac{g(x+he_k) - g(x)}{h}$

$= -\int \frac{f(x-he_k)-f(x)}{-h}\, g(x)$

$= -\int D_k^{-h} f \, g$

$fg(x+he_k) - f(x)$

$= f(x+he_k)\left[g(x+he_k) - g(x)\right]$

$+ g(x)\left[f(x+he_k) - f(x)\right]$

$\left[v^h = v(x+he_k) \right]$

Hence we can write :

$$A = \sum_{i,j}\int_U \underbrace{a^{ij,h}(D_k^h u_{x_i})(D_k^h u_{x_j})\zeta^2\, dx}_{=:A_1} \; +$$

$$+ \underbrace{\sum_{i,j}\int_U \Big[a^{ij,h}(D_k^h u_{x_i})(D_k^h u)2\zeta\zeta_{x_j} + (D_k^h a^{ij})u_{x_i} D_k^h u_{x_j}\zeta^2 + (D_k^h a^{ij}) u_{x_i}(D_k^h u) 2\zeta\zeta_{x_j}\Big]\, dx}_{=A_2}$$

Unif. ellipt. $\Rightarrow$

$$A_1 \geq \theta \int_U \zeta^2 |D_k^h Du|^2 dx$$

Also $(\zeta^2 \leq \zeta)$

$$|A_2| \leq C\int_U \zeta|D_k^h Du||D_k^h u| + \zeta|D_k^h Du||Du| + \zeta|D_k^h u||Du|\,dx$$

ε- Cauchy $\Rightarrow$

$$|A_2| \leq \varepsilon\int_U \zeta^2|D_k^h Du|^2 dx + \frac{C}{\varepsilon}\int_U |D_k^h u|^2 + |Du|^2 dx$$

Now we need a lemma :

Lemma 1 $1 \leq p < \infty$, $u \in W^{1,p}(U)$. Then $\forall\, V \subset\subset U$:

$$\|D^h u\|_{L^p(V)} \leq C\|Du\|_{L^p(U)}$$

$\forall\, h,\; 0 < |h| < \operatorname{dist}(V, \partial U)/2$.

We will prove this later ;

By lemma 1, choosing $\varepsilon = \theta/2$

$$|A_2| \leq \frac{\theta}{2}\int_U \zeta^2|D_k^h Du|^2 dx + C\int_U |Du|^2 dx$$

$$\therefore\; A \geq \frac{\theta}{2}\int_U \zeta^2 |D_k^h Du|^2 dx - C\int_U |Du|^2 dx. \quad (\ddot\smile)$$

<u>Step V</u> Using similar ideas one can prove

$$|B| \le \frac{\theta}{4} \int_U \sum |D_k^h Du|^2 dx + C \int_U f^2 + u^2 + |Du|^2 dx \qquad (\widetilde{\ddot\smile})$$

Step $\underline{VI}$: Now $A = B$ & $(\ddot\smile)$, $(\widetilde{\ddot\smile})$ give

$$\frac{\theta}{2} \int_U \sum |D_k^h Du|^2 dx - C\int_U |Du|^2 dx \le A = B \le \frac{\theta}{4} \int_U \sum |D_k^h Du|^2 dx + C \int_U f^2 + u^2 + |Du|^2 dx$$

Hence

$$\frac{\theta}{4} \int_V |D_k^h Du|^2 dx \le \frac{\theta}{4} \int_U \sum |D_k^h Du|^2 dx \le C \int_U f^2 + u^2 + |Du|^2 dx .$$

Now we need yet another lemma : $\qquad (V \subset\subset U)$

<u>Lemma 2.</u> Assume $1 < p < \infty$, $u \in L^p(V)$ and $\exists\, C \ge 0$ s.t.

$$\|D^h u\|_{L^p(V)} \le C \qquad \forall\ 0 < |h| < \tfrac{1}{2} \operatorname{dist}(V, \partial U).$$

Then

$$u \in W^{1,p}(V), \quad \|Du\|_{L^p(V)} \le C .$$

Again proof of this later.

Now

Lemma 2 $\Rightarrow$ $u \in H^2(V)$,

$$\|u\|_{H^2(V)} \le C \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right)$$

To get the desired upper bound i.e. how to replace $\|u\|_{H^1(U)}$ by $\|u\|_{L^2(U)}$, see [Evans] □ / mod Lemmas 1 & 2

<u>Pf. of Lemma 1</u>

Let $1 \le p < \infty$ and assume 1^{st} that u is C^∞.

Then $\forall\, x \in V$, $0 < |h| < \operatorname{dist}(V, \partial U)$, we have

$$u(x + he^i) - u(x) = h \int_0^1 u_{x_i}(x + the^i)\, dt$$

$\Rightarrow$

$$|u(x+he^i) - u(x)| \le h \int_0^1 |Du(x + the^i)|\, dt$$

$\therefore$

$$\int_V |D^h u|^p dx \le C \sum_{i=1}^n \int_V \int_0^1 |Du(x + the^i)|^p\, dt\, dx$$

Fubini

$\Rightarrow$ $\qquad$ General case by approximation

$$\int_V |D^h u|^p dx \le C \int_U |Du|^p dx \ \square$$

<u>Pf. of Lemma 2</u>

Choose $\phi \in C_0^\infty(V)$, $|h|$ small enough. Then

$$\int_V u(x) \left[\frac{\phi(x + he_i) - \phi(x)}{h} \right] dx \overset{|h| \text{ small}}{=} - \int_V \left[\frac{u(x - he_i) - u(x)}{h} \right] \phi(x)\, dx$$

$\Leftrightarrow$

$$\int_V u\, D_{e_i}^h \phi\, dx = - \int_V (D_{e_i}^{-h} u)\, \phi\, dx$$

Now
$$\sup_{|h|<h_0} \|D_i^{-h} u\|_{L^p(V)} < C$$

i.e. $1<p<\infty$ + Banach-Alaoglu $\Rightarrow$ $\exists\ v_i \in L^p(V)$ and a subsequence h_k s.t.
$$w-\lim_{k\to\infty} D_i^{-h_k} u = v_i$$

Hence
$$\int_V u\phi_{x_i}\,dx = \int_U u\phi_{x_i} = \lim_{h_k\to 0} \int_U u D_i^{h_k}\phi\,dx$$
$$= -\lim_{h_k\to 0} \int_V (D_i^{-h_k} u)\phi\,dx = -\int_V v_i\,\phi\,dx = -\int_U v_i\phi\,dx$$

$\therefore$ $u_{x_i} = \underset{\in L^p(V)}{v_i}$ weakly $\Rightarrow u \in W^{1,p}(V)$. $\square$

One can iterate this to get

<u>Thm 2</u> Assume

$a^{ij}, b^i, c \in C^{m+1}(U)$.

$f \in H^m(U)$.

If u solves $Lu=f$ weakly, then

$u \in H^{m+2}_{loc}(U)$

For proof see [Evans].

<u>Note:</u> If $m=\infty$, then by embedding we have the "elliptic regularity": $a^{ij}, b^i, c \in C^\infty$, $f \in C^\infty$ & $Lu=f$ in U weakly $\Rightarrow u \in C^\infty(U)$.

2.8. Boundary regularity

<u>Thm.</u> Assume

$a^{ij} \in C^1(\bar U)$, $b^i, c \in L^\infty(U)$

and $f \in L^2(U)$.

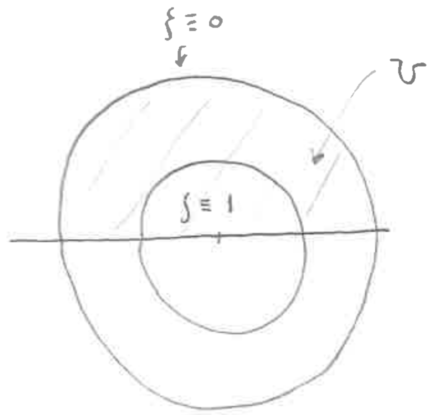
If $u \in H^1_0(U)$ is a weak solution of (L str. elliptic)
$$\begin{cases} Lu = f & \text{in } U \\ u|_{\partial U} = 0 \end{cases}$$
and ∂U is C^2, then $u \in H^2(U)$ and
$$\|u\|_{H^2(U)} \le C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)}).$$

Pf. Step I Let's 1st assume
$$U = B(0,1) \cap \mathbb{R}^n_+ \quad \text{(even though this is not a } C^2 \text{ domain (:))}$$

Let
$$V = B(0,1/2) \cap \mathbb{R}^n_+.$$

Choose $\zeta \in C^\infty(\mathbb{R}^n)$ s.t.

$$\begin{cases} \zeta = 1 \text{ on } B(0,\tfrac12), \ \zeta \equiv 0 \text{ in } \mathbb{R}^n \setminus \overline{B(0,1)} \\ 0 \le \zeta \le 1 \end{cases}$$



Step II Now as before.

(x) $B[u,v] = \int f v \quad \forall v \in H_0^1(V)$

$\Rightarrow$

$$\sum_{i,j} \int_V a^{ij} u_{x_i} v_{x_j} \, dx = \int \tilde{f} v \, dx,$$

$$\tilde{f} = f - \sum_i b^i u_{x_i} - cu.$$

Step III Choose $h>0$ small enough, $k \in \{1, \dots, n-1\}$, $\leftarrow$ ∇_0 i.e. only tangential directions!

let

$$v = -D_k^{-h}(\zeta^2 D_k^h u).$$

Note

$$v(x) = -\frac{1}{h} D_k^{-h}\left[\zeta^2(x)\left(u(x+he_k) - u(x)\right)\right]$$

$$= \frac{1}{h^2}\left[\zeta^2(x-he^k)\left(u(x) - u(x-he_k)\right) - \zeta^2(x)\left(u(x+he_k) - u(x)\right)\right]$$

$\in H_0^1(V)$ | Does Not work for the normal direction!

As before we choose v as above, subs. this to (x), and write it as

$$A = B,$$

$$A = \sum_{i,j} \int_V a^{ij} u_{x_i} v_{x_j} \, dx$$

$$B = \int_V \tilde{f} v \, dx$$

and as before we get an estimate

$$A \ge \frac{\theta}{2} \int_V \zeta^2 |D_k^h Du|^2 dx - C \int_V |Du|^2 dx$$

$$B \le \frac{\theta}{4} \int_V \zeta^2 |D_k^h Du|^2 dx + C\left(\int_V f^2 + u^2 + |Du|^2 dx\right)$$

i.e. we get

$$\int_V |D_k^h Du|^2 dx \le C\left(\|f\|_{\Omega} + \|u\|_{H^1}\right)$$

$\Rightarrow \quad u_{x_k} \in H^1(V), \quad k \in \{1, \dots, n-1\}.$

Step IV So, the boundary cond. & smoothness are needed to control the $u_{x_n x_n}$ term!

Now

$$u \in H_0^1(V), \quad a^{ij} \in C^1(\bar{V}), \ b^i, c \in L^\infty$$

$\Rightarrow \quad Lu = f$ a.e. in V.

controlling $u_{x_n x_n}$

For this one can use the equation:

$$a^{nn}(x) u_{x_n x_n} = - \sum_{\substack{i,j=1 \\ i+j \neq 2n}}^{n} a^{ij}(x) u_{x_i x_j} + \underbrace{\sum \tilde{b}^i u_{x_i}(x_i) + cu - f}_{\text{contains derivatives of } a^{i}} .$$

Uniform ellipticity:

$$\sum_{i,j} a^{ij}(x) \xi_i \xi_j \geq \theta |\xi|^2 .$$

Choose $\xi = (0, \dots, 1)$. Hence $a_{nn}(x) \geq \theta > 0$

and hence we get a.e. $x \in V$:

$$|u_{x_n x_n}(x)| \leq C \Big(\sum_{\substack{i+j=1 \\ i+j < 2n}}^{n} |u_{x_i x_j}(x)| + |Du(x)| + |u(x)| + |f(x)| \Big)$$

$\leadsto$ we get an L^2-bound for $u_{x_n x_n}$!

<u>Last step</u> For a general U use a partition of unity & straight. of bndry to find a map $\Phi_i : V_i \longrightarrow U$, $\{V_i\}$ an open cover of ∂U. Essential is to check that L is also elliptic in these new coordinates:

$$x = \Phi_i(y), \quad y = \Psi_i(x)$$

$$Lu(y) = \sum a^{ij}(y) \frac{\partial^2 u}{\partial y_i \partial y_j} + \text{lower order}$$

$$\begin{tikzcd} V_i \arrow[r, "\Phi_i"] \arrow[d, "u"'] & U \arrow[dl, "\hat{u}"] \\ \mathbb{R} & \end{tikzcd}$$

$$\tilde{u} = u \circ (\Phi_i^{-1}) = u \circ \Psi_i , \quad \Psi_i = \Phi_i^{-1}$$

$$\frac{\partial \tilde{u}}{\partial x_k} = \sum_{\ell} \frac{\partial u}{\partial y_\ell} \frac{\partial y_\ell}{\partial x_k} = (\Psi')^t_k Du$$

$$\therefore D_x \tilde{u} = (\Psi_i')^t D_y u \qquad D_y u = (\Psi_i'^t)^{-1} D_x \tilde{u}$$
$$= (\Phi_i')^t D_x \tilde{u}$$

Hence

$$Lu(y) = \tilde{L} \hat{u}(x),$$

and

$$B_L[u, v] = \int_{V_i} \langle A(y) D_y u, D_y v \rangle dy + \text{lower order}$$

$$= \int_{V} \langle A \circ \Psi_i(x) (\Phi_i')^t(x) D_x \hat{u}, (\Phi_i')^t(x) D_x \hat{v} \rangle |\Phi'|^{-1} dx + \text{lower order}$$

$$= \int_{V} \langle \tilde{A}(x), D_x \tilde{u}, D_x \hat{v} \rangle + \text{lower order},$$

where

$$\tilde{A}(x) = \frac{(\Phi_i')(x) A(x) (\Phi_i')^t(x)}{|\Phi'(x)|}$$

(1)

2.3. Maximum principles

As we saw last time – and as it's well known from intro to PDE's course, harmonic functions satisfy the weak max. principle:

$$\begin{cases} U \Subset \mathbb{R}^n \text{ bnd}, & u \in C^2(U) \cap C(\bar{U}), \\ \Delta u = 0 \text{ in } U \end{cases}$$

$\Rightarrow \quad \max_{\bar{U}} u = \max_{\partial U} u$ (and hence also for min)

Now we prove this for more general elliptic ops

$$Lu = -\sum a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b^i(x) \frac{\partial u}{\partial x_i} + c(x) u .$$

Note that on purpose we do not write this in divergence form.

We assume

$$a^{ij}, b^i, c \in L^\infty(U) \cap C(U), \quad a^{ij} = a^{ji}$$

$$U \Subset \mathbb{R}^n \text{ bnd}, \quad a^{ij} \text{ unif. elliptic.}$$

Thm. (Weak Maximum Princ.) Let $u \in C^2(U) \cap C(\bar{U})$.

Assume $c \equiv 0$ (Think what could go wrong?)

(i) If $Lu \leq 0$ in U (i.e. u is a subsolution)

then

$$\max_{\bar{U}} u = \max_{\partial U} u .$$

(ii) If $Lu \geq 0$ in U (i.e. u is a supersolution), then

$$\min_{\bar{U}} u = \min_{\partial U} u .$$

Pf. Step I. Let's 1st assume (like for $L = -\Delta$),

$$Lu < 0 \text{ in } U,$$

and $\exists\, x_0 \in U$ s.t.

$$u(x_0) = \max_{\bar{U}} u .$$

Then

$$\begin{cases} \nabla u(x_0) = 0 \\ D^2 u(x_0) \leq 0 \quad \text{i.e. the Hessian is neg. definite)} \end{cases}$$

Step II $A(x_0) = (a^{ij}(x_0))$ symmetric $\Rightarrow \exists\, O \in ON(n)$ s.t.

$$A(x_0) = O^t \Lambda O, \quad \Lambda = (\lambda_1, \dots, \lambda_n), \quad \lambda_1, \dots, \lambda_n > 0 .$$

Let

$$y - y_0 = O(x - x_0),$$

Then

$$\nabla_x u = O^t \nabla_y u \qquad \frac{\partial u}{\partial x_i} = \sum_k \frac{\partial u}{\partial y_k} o^{ki}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{k,l} o^{ki} o^{lj} \frac{\partial^2 u}{\partial y_i \partial y_j}$$

$$\Rightarrow \sum_{i,j} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = \sum_{i,j,k,l} o^{ki} a^{ij}(x) o^{lj} \frac{\partial^2 u}{\partial y_k \partial y_l}$$

$\Rightarrow 0 > Lu(x_0) = \sum \lambda_\nu \frac{\partial^2 u}{\partial y_\nu^2} > 0$ ↯. Hence $\nexists$ local max in U.

Step III Assume now $Lu \leq 0$. Let

$$u^\varepsilon(x) := u(x) + \underbrace{\varepsilon e^{\lambda x_1}}_{>0}, \quad \varepsilon > 0$$

Then

$$Lu^\varepsilon(x) = Lu(x) - \varepsilon\left[\lambda^2 a^{11}(x) + \lambda b^1(x)\right] e^{\lambda x_1} < 0 \text{ if } \lambda > 0 \text{ large enough}$$

Note: Ellipticity $\Rightarrow$ $a^{11}(x) \geq \theta > 0$ in U.

Then

$$\max_{\bar U} u \leq \max_{\bar U} u_\varepsilon = \max_{\partial U} u_\varepsilon \leq \max_{\partial U} u + C\varepsilon$$

Taking $\varepsilon \to 0$ gives the claim. $\square$

How to prove strong max. principles we leave ~~this~~ as HW (Read & use Hopf-lemma).

III VARIATIONAL CALCULUS

3.1. Motivation & Intro.

Example 1 Let's consider a functional (note that this is generally nonlinear)

(*) $I[w] := \frac{1}{2}\int_U |Dw|^2 \, dx$, $U \Subset \mathbb{R}^n$

where $w : U \to \mathbb{R}$ smooth enough and

(**) $w|_{\partial U} = g$.

Assume w minimizes $I[w]$ for all w satisfying (**).
Let $v \in C_0^\infty(U)$, and (v real valued)

$$i(t) := I[w + tv].$$

Then

$$w \text{ minimizer} \Rightarrow i'(t)\big|_{t=0} = 0.$$

Now

$$I[w+tv] - I[w] = \frac{1}{2}\int_U \langle D(w+tv), D(w+tv)\rangle - \langle Dw, Dw\rangle \, dx$$

$$= \frac{1}{2}\int_U 2t\langle Dw, Dv\rangle + t^2|Dv|^2 \, dx$$

$$\Rightarrow \quad i'(0) = \int_U \langle Dw, Dv\rangle \, dx = -\int_U \Delta w \cdot v \, dx = 0 \quad \forall v \in C_0^\infty$$

$$\Rightarrow \quad \Delta w = 0 \text{ in } U.$$

Hence, assuming

(i) w is the minimizer of $I[w]$ under constraint $w|_{\partial U} = g$

(ii) w is smooth enough (i.e. $w \in C^2(U)$ or $w \in H^1(U)$ for weak sols)

we see that w is the unique solution of the Dirichlet problem

$$\begin{cases} \Delta w = 0 \text{ in } U \\ w|_{\partial U} = g \end{cases}$$

We aim to generalize this idea to more general functionals. It is often easy to show (under suitable assumptions of course) that functionals have (even unique) minimizers. In some cases this yields solutions to really hard PDE's.

3.2. The 1st variation & Euler-Lagrange equations

Assume $U \Subset \mathbb{R}^n$ bnd, with C^∞ boundary.
Let

$$L : \mathbb{R}^n \times \mathbb{R} \times \bar{U} \longrightarrow \mathbb{R}$$

be a smooth function. We call L the Lagrangian.

We write

$$L(p,z,x) = L(p_1,\dots,p_n, z, x_1,\dots,x_n),$$

$$\left(\begin{array}{l} p = \text{"momentum"} \\ z = \text{value of } w \\ x = \text{position} \end{array}\right)$$

$$D_p L = \left(\frac{\partial L}{\partial p_1},\dots,\frac{\partial L}{\partial p_n}\right) = \frac{\partial L}{\partial p}$$

$$D_z L = \frac{\partial L}{\partial z}$$

$$D_x L = \left(\frac{\partial L}{\partial x_1},\dots,\frac{\partial L}{\partial x_n}\right) = \frac{\partial L}{\partial x}.$$

Let

$$\left[\; I[w] = \int_U L(Dw(x), w(x), x)\,dx, \quad \begin{cases} w : \bar U \to \mathbb{R} \quad \text{smooth} \\ w|_{\partial U} = g \end{cases} \;\right]$$

Choose any $v \in C_0^\infty(U)$, and let

$$i(\tau) = I[u + \tau v]$$

$$= \int_U L(Du + tDv, w + tv, x)\,dx.$$

Then

$$i'(\tau) = \int_U \sum_{i=1}^n \frac{\partial L}{\partial p_i}(Du + tDv, u + tv, x)\underbrace{}_{\frac{\partial v}{\partial x_i}} + \frac{\partial L}{\partial z}(Du + tDv, u + tv, x)\,v\,dx$$

$$\overset{v\in C_0^\infty}{=} \int_U \left[\sum_{i=1}^n -\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial p_i}(Du + tDv, u + tv, x)\right) + \frac{\partial L}{\partial z}(Du + tDv, u + tv, x)\right] v\,dx$$

So if u is a minimizer $\Rightarrow$

$$0 = i'(0) = \int_U \left[-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial p_i}(Du,u,x)\right) + \frac{\partial L}{\partial z}(Du,u,x)\right] v\,dx \quad \forall v \in C_0^\infty(U)$$

$\Rightarrow$ u solves the (generally nonlinear) PDE

$$\begin{cases} -\sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{\partial L}{\partial p_i}(Du,u,x) + \frac{\partial L}{\partial z}(Du,u,x) = 0 \text{ in } U \\ u|_{\partial U} = g. \end{cases}$$

[The Euler-Lagrange eqn associated to I]

<u>Example 2</u> $\quad L(p,z,x) = \frac{1}{2}|p|^2$ gives

$$I[u] = \frac{1}{2}\int_U |Du|^2\,dx, \quad u|_{\partial U} = g$$

i.e. the Euler-Lagrange eqn is

$$\begin{cases} \Delta u = 0 \text{ in } U \\ u|_{\partial U} = g. \end{cases}$$

This is known as the Dirichlet's principle.

<u>Ex. 3</u> (Generalized Dirichlet's principle)

$$L(p,z,x) = \frac{1}{2}\sum_{i,j} a^{ij} p_i p_j - z f(x) = \frac{1}{2}\langle p, Ap\rangle - z f(x)$$

$$A(x) = (a^{ij}(x)).$$

Then, if $A^t = A$,

$$L_{p_i} = \sum_j a^{ij}(x) p_j = A(x) p,$$

so the Euler-Lagrange eqn associated to functional

$$I[w] = \int_U \frac{1}{2}\sum_{i,j=1}^n a^{ij}(x) \frac{\partial w}{\partial x_i}\frac{\partial w}{\partial x_j} - w f(x)\, dx$$

is

$$-\nabla \cdot A(x) \nabla u = f \quad \text{in } U$$

Ex. 4 (Nonlinear Poisson eqn)

Let $f: \mathbb{R} \to \mathbb{R}$ be smooth,

$$F(z) = \int_0^z f(y)\,dy$$

its integral function ($F' = f$). Let

$$I[w] = \int_U \tfrac{1}{2}|Dw|^2 - F(w)\, dx.$$

Hence

$$L(p,z,x) = \tfrac{1}{2}|p|^2 - F(z),$$

$$\frac{\partial L}{\partial p} = p, \quad \frac{\partial L}{\partial z} = -F'(z) = f(z),$$

i.e. the corresponding Euler-Lagrange eqn is

$$-\Delta u = f(u) \quad \text{in } U$$

Ex. 4 (Minimal Surfaces)

Let $L(p,z,x) = (1+|p|^2)^{1/2}$

i.e.

$$I[w] = \int_U (1+|Dw|^2)^{1/2}\, dx \qquad (= \text{area of the graph of } w \text{ over } U)$$

Then

$$\frac{\partial L}{\partial p} = \frac{1}{2}(1+|p|^2)^{-1/2} \cdot 2p,$$

i.e. the corresponding Euler-Lagrange is the minimal surface equation

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial u/\partial x_i}{(1+|\nabla u|^2)^{1/2}}\right) = 0 \quad \text{in } U$$

This is hard! (☺)

$$= \nabla \cdot \left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right) = n \cdot \text{mean curvature of graph of } u.$$

3.3. Second variation

To differentiate a local minima from local maxima, we compute the second variation of $I[\cdot]$ at u.

Now

$$i'(\tau) = \int_U \sum_{i=1}^n -\frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial p_i}(Du+\tau Dv, u+\tau v, x)\right) + \frac{\partial L}{\partial z}(Du+\tau Dv, u+\tau v, x)\Bigg)\, v\, dx$$

easier
$$= \int_U \sum_{i=1}^n \frac{\partial L}{\partial p_i}(Du+\tau Dv, u+\tau v, x)\frac{\partial v}{\partial x_i} + \frac{\partial L}{\partial z}(Du+\tau Dv, u+\tau v, x)\, v\, dx$$

Hence

$$i''(\tau) = \int_U \sum_{i,j} \frac{\partial^2 L}{\partial p_i \partial p_j}(Du+\tau Dv, u+\tau v, x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + 2\sum_i \frac{\partial^2 L}{\partial p_i \partial z}(Du+\tau Dv, u+\tau v, x)\frac{\partial v}{\partial x_i} v$$
$$+ \frac{\partial^2 L}{\partial z^2}(Du+\tau v, u+\tau v, x)\, v^2\, dx .$$

Hence at a minimum

$$(☺)\quad 0 \le i''(0) = \int_U \sum_{i,j} \frac{\partial^2 L}{\partial p_i \partial p_j}(Du,u,x)\frac{\partial v}{\partial x_i}\frac{\partial v}{\partial x_j} + 2\sum_i \frac{\partial^2 L}{\partial p_i \partial z}(Du,u,x)\frac{\partial v}{\partial x_i} v$$
$$+ \frac{\partial^2 L}{\partial z^2}(Du,u,x)\, v^2\, dx \qquad \forall\, v \in C_0^\infty(U) .$$

Now we need:

__Prop.__ $U \in \mathbb{R}^n$ bnd; ∂U is C^1. Then $u \in W^{1,\infty}(U) \iff u$ Lip-cont. in U.

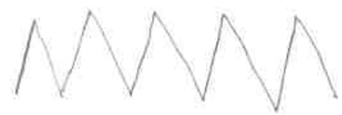
Also, if $u_\varepsilon = u * \eta_\varepsilon$, η_ε the standard mollifier, then
$$\begin{cases} u_\varepsilon \to u \text{ unif.} \\ \|Du_\varepsilon\|_{L^\infty} \le \|Du\|_{L^\infty} \end{cases}$$
and (U bnd) $u_\varepsilon \to u$ in $W^{1,p}(U)$ $\forall\, 1 \le p < \infty$.

Pf. Later / if even then. See [Evans; sect. 5.8.(b)]

This gives now the possibility – via approximation – to replace v in (☺) with any Lipschitz-function. ($=0$ on ∂U)

We will use the following:

Fix $\xi \in \mathbb{R}^n$. Let
$$v(x) = \varepsilon \rho\left(\frac{\langle x, \xi\rangle}{\varepsilon}\right) \zeta(x), \quad x \in U$$
where $\zeta \in C_0^\infty(U)$ and ρ is 1-periodic $\mathbb{R} \to \mathbb{R}$,
$$\rho(x) = \begin{cases} x, & 0 \le x \le 1/2 \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}$$



$(\rho(x+1) = \rho(x))$.

Then $|\rho'| = 1$ a.e. and
$$\frac{\partial v}{\partial x_i}(x) = \rho'\left(\frac{\langle x, \xi\rangle}{\varepsilon}\right)\xi_i \zeta(x) + O(\varepsilon)$$
and hence (☺) gives
$$0 \le \int_U \sum_{i,j} \frac{\partial^2 L}{\partial p_i \partial p_j}(Du,u,x)\,\xi_i \xi_j \underbrace{\rho'\left(\frac{\langle x, \xi\rangle}{\varepsilon}\right)^2}_{\equiv 1 \text{ a.e.}} \zeta^2(x)\, dx + O(\varepsilon)$$

So letting $\varepsilon \to 0$ we get
$$\int_U \sum_{i,j} \frac{\partial^2 L}{\partial p_i \partial p_j}(Du,u,x)\,\xi_i \xi_j\, \zeta^2(x)\, dx \ge 0 \quad \forall\, \zeta \in C_0^\infty(U)$$

$\Rightarrow$
$$\sum_{i,j} \frac{\partial^2 L}{\partial p_i \partial p_j}(Du,u,x)\,\xi_i \xi_j \ge 0 \quad \forall\, \xi \in \mathbb{R}^n, x \in U .$$
("a convexity for L")

All of this generalizes also to systems:

Assume

$$L : \mathbb{M}^{m\times n} \times \mathbb{R}^m \times \bar{U} \longrightarrow \mathbb{R} \quad \text{smooth}$$

$\mathbb{M}^{m\times n}$ = v.space of real $m \times n$ matrices

$$P = \begin{pmatrix} p^1 \\ \vdots \\ p^m \end{pmatrix}, \quad \text{where} \quad p^k = (p^k_1, \dots, p^k_n)$$

and we take

$$P = Dw \; ; \quad w : \bar{U} \longrightarrow \mathbb{R}^m \text{ smooth}, \quad w = \begin{pmatrix} w^1 \\ \vdots \\ w^m \end{pmatrix},$$

$$Dw = \begin{pmatrix} \frac{\partial w^1}{\partial x_1} & \cdots & \frac{\partial w^1}{\partial x_n} \\ \vdots & & \\ \frac{\partial w^m}{\partial x_1} & \cdots & \frac{\partial w^m}{\partial x_n} \end{pmatrix} \qquad z = \begin{pmatrix} z^1 \\ \vdots \\ z^m \end{pmatrix}$$

Proceedin as before we see that a (local) minimizer u of the functional

$$I[w] = \int_U L(Dw, w, x)\, dx$$

satisfies the <u>system</u> of E-L equs

$$(EL)_k \quad -\sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial L}{\partial p^k_i}(Du, u, x)\right) + \frac{\partial L}{\partial z^k}(Du, u, x) = 0, \quad k = 1, \dots, m.$$

3.4. <u>Null Lagrangians</u>

<u>Def.</u> A Lagrangian L is a null-Lagrangian if the E-L system $(E\text{-}L)_k$ is satisfied by any smooth u.

<u>Thm.</u> Ass. L a null-Lagrangian,

$$I[w] = \int_U L(Dw, w, x)\, dx.$$

If $u, \hat{u} \in C^2(\bar{U}, \mathbb{R}^m)$ s.t. $u = \hat{u}$ on ∂U then

$$I[u] = I[\hat{u}].$$

Pf. Let

$$i(\tau) = I\big[\overbrace{\tau u - (1-\tau)\hat{u}}^{=:\, w_\tau}\big], \quad 0 \le \tau \le 1.$$

Then

$$i'(\tau) = \int_U \sum_{i=1}^n \sum_{k=1}^m \frac{\partial L}{\partial p^k_i}(Dw_\tau, w_\tau, x)\frac{\partial}{\partial x_i}(u^k - \hat{u}^k)$$

$$+ \sum_{k=1}^m \frac{\partial L}{\partial z^k}(Dw_\tau, w_\tau, x)\Big)(u^k - \hat{u}^k)\, dx$$

$$\overset{u = \hat{u} \text{ on } \partial U}{=} \int_U \Big[-\sum_i \sum_k \frac{\partial}{\partial x_i}\Big(\frac{\partial L}{\partial p^k_i}(Dw_\tau, w_\tau, x)\Big) + \frac{\partial L}{\partial z^k}(Dw_\tau, w_\tau, x)\Big](u^k - \hat{u}^k)$$

$$= 0$$

since L a null Lagrangian $\Rightarrow$ $i(\tau)$ const $\Rightarrow$ claim. □

Ex. $(k=1)$ $L(p,z,x) = ap + \ell(x)$, $a : \bar{U} \longrightarrow \mathbb{R}$,

is a Null-Lagrangian: The E-L equ's read

$$-\sum \frac{\partial}{\partial x_i}\underbrace{\left(\underbrace{\frac{\partial L}{\partial p_i}(Du,u,x)}_{\equiv\, const}\right)}_{\equiv\, 0} + \underbrace{\frac{\partial L}{\partial z}(Du,u,x)}_{\equiv\, 0} = 0 .$$

Let $A \in \mathbb{M}^{n\times n}$, then

$$\operatorname{cof} A = \left((\operatorname{cof} A)^k_i \right), \quad (\operatorname{cof} A)^k_i = (-1)^{i+k} \det A^k_i ,$$

$A^k_i = (n-1)\times(n-1)$ matrix obtd from A by deleting row k & column i.

Hence

$$\det A = \sum_{i=1}^{n} a^k_i (\operatorname{cof} A)^k_i = \sum_{k=1}^{n} a^k_i (\operatorname{cof} A)^k_i \quad \forall\, i, k .$$

Lemma (Divergence free rows). Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be smooth.

Then

$$\sum_{i=1}^{n} (\operatorname{cof} Du)^k_{i, x_i} = 0 \quad \forall\, k \in \{1, \dots, n\}$$

i.e. the cofactor matrix has divergence free rows.

Pf. Now

$$(*) \qquad P^T(\operatorname{cof} P) = (\det P)\, I \iff (\operatorname{cof} P)^T P = (\det P)\, I$$

$$\iff P^{-1} = \frac{(\operatorname{cof} P)^T}{|\det P|} \text{ if } P \neq 0 \ \left(\text{Kramer's rule}\right)$$

i.e.

$$(**) \qquad \sum_{k=1}^{n} p^k_i (\operatorname{cof} P)^k_j = (\det P)\, \delta_{ij} .$$

Hence (take $i = j = m$ above)

$$\frac{\partial \det P}{\partial p^k_m} = (\operatorname{cof} P)^k_m \qquad (\text{since } \operatorname{cof} P \text{ does not contain } p^k_m !)$$

Substitute into $(**)$ $P = Du$: we have

$$\sum_{k=1}^{n} \frac{\partial u^k}{\partial x_i} (\operatorname{cof} Du)^k_j = (\det Du)\, \delta_{ij}$$

Now

$$\frac{\partial}{\partial x_j} \sum_k \frac{\partial u^k}{\partial x_i} (\operatorname{cof} Du)^k_j = \sum_k \frac{\partial^2 u^k}{\partial x_i \partial x_j} (\operatorname{cof} Du)^k_j + \frac{\partial u^k}{\partial x_i} \frac{\partial (\operatorname{cof} Du)^k_j}{\partial x_j}$$

and

$$\frac{\partial}{\partial x_j} (\det Du)\, \delta_{ij} = \sum_{m,k} \frac{\partial (\det Du)}{\partial p^k_m} \frac{\partial p^k_m}{\partial x_j} \delta_{ij}$$

$$= \sum_{m,k} \frac{\partial (\det Du)}{\partial p^k_m} \frac{\partial^2 u^k}{\partial x_j \partial x_m} \delta_{ij} = \sum_{m,k} (\operatorname{cof} Du)^k_m \frac{\partial^2 u^k}{\partial x_j \partial x_m} \delta_{ij},$$

and hence summing over j we get

$$\sum_{j,m,k} (\operatorname{cof} Du)^k_m \frac{\partial^2 u^k}{\partial x_j \partial x_m} \delta_{ij} = \cancel{\sum_{j,k} \frac{\partial^2 u^k}{\partial x_i \partial x_j} (\operatorname{cof} Du)^k_j} + \frac{\partial u^k}{\partial x_i} \frac{\partial (\operatorname{cof} Du)^k_j}{\partial x_j}$$

$$\cancel{\sum_{m,k} (\operatorname{cof} Du)^k_m \frac{\partial^2 u^k}{\partial x_i \partial x_m}}$$

$$\Leftrightarrow \quad \sum_{j,k} \frac{\partial u^k}{\partial x_i} \frac{\partial (\operatorname{cof} Du)^k_j}{\partial x_j} = 0, \quad i = 1, \dots, n.$$

If $\det Du(x_0) \neq 0$ i.e. rows $\partial u^k / \partial x_i$ are lin. indep $\Rightarrow$

$$\sum_j \frac{\partial (\operatorname{cof} Du)^k_j}{\partial x_j} = 0$$

i.e. claim. If $\det Du(x_0) = 0$, choose $\varepsilon_0 > 0$ so small that

$$\det(Du(x_0) + \varepsilon \mathbb{I}) \neq 0, \quad 0 < \varepsilon < \varepsilon_0 \quad \text{(eigenvalues are isolated)}$$

Then

$$\sum_j \frac{\partial (\operatorname{cof} D(u + \varepsilon \mathbb{I}))^k_j}{\partial x_j} = 0, \quad 0 < \varepsilon < \varepsilon_0$$

$$\Big\downarrow \ \varepsilon \to 0$$

$$\sum_j \frac{\partial (\operatorname{cof} Du)^k_j}{\partial x_j}, \quad \square$$

Thm. $L(P) = \det P$, $P \in M^{n\times n}$ is a null-Lagrangian.

Pf. Let u be smooth. E-L eqns for $\det P$ are

$$\sum_i \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial p_i^k}(\det u)\right) = 0.$$

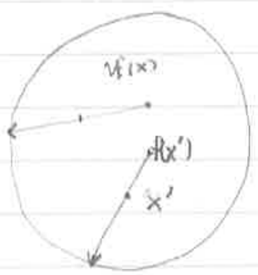
But $\frac{\partial}{\partial p_i^k}(\det u) = (\mathrm{cof}\, Du)_i^k$ and hence the claim follows from the previous result. □

3.5. Brouwer's fixed point Thm.

Thm (Brouwer) Assume $u : \overline{B(0,1)} \to \overline{B(0,1)}$ is continuous. Then u has a fixed pt.

Pf using Alg. Top: ($n=0$ trivial) Assume $n>0$, and u has no fixed pt.

Can also use homotopy groups



If $x \neq u(x)\ \forall x$, let

$g(x)$ = the unique bnd pt on the ray from $f(x)$ to x

g cont. and $g(x) = x$ if $x \in \partial B(0,1)$

$\therefore g$ a retraction.

Uses the long exact homology sequence.

Then

$$\underbrace{H^1(B(0,1))}_{=0} \cong \underbrace{H^1(\partial B(0,1))}_{\cong \mathbb{Z}} \oplus H^1(B(0,1), \partial B(0,1)) \quad \text{↯}. \ \square$$

cont.

Tar
$f: B^n \to S^{n-1}$ retr.
$g: S^{n-1} \hookrightarrow B^n$

— Analytic pf: $B = \overline{B(0,1)}$

Step 1 We claim that $\nexists$ smooth retraction $w : B \to \partial B$

Suppose there is. Let $\tilde{w} : B \to B$, $\tilde{w} = id$ ($\tilde{w}(x) = x$).

Then $w = \tilde{w}$ on ∂B and since $\det P$ is a null Lagrangian,

$f_\#: \pi^1(B^n, x_0) \to \pi^1(S^{n-1}, x_0)$
$g_\#: \pi^1(S^{n-1}, x_0) \to \pi^1(B^n, x_0) = \{0\}$

$g^\# f^\# = (g\circ f)^\# = \cancel{0 = g_\# f_\# = (g\circ f)^\#} =$ $\quad 0 = f_\# g_\# = (f\circ g)_\# = id_\# = 1_{\pi^{n-1}(S^{n-1}, x_0)}$ ↯.

$\Rightarrow$

$$\int_B \det Dw\,dx = \int_B \det \underbrace{D\tilde{w}}_{=id}\,dx = m(B).$$

On the other hand, $|w(x)| = 1 \quad \forall x \in B \iff \langle w, w\rangle = 1$

$\Rightarrow$

$$(Dw)^T w = 0$$

Hence 0 is an eigenvalue of $(Dw)^T \quad \forall x \in B$ (Note: $|w(x)| = 1 \quad \forall x \Rightarrow w(x) \neq 0$). Hence $\det Dw = 0$ and

$$\int_B \det Dw\,dx = 0 \,!$$

<u>Step 2.</u> Need we show that $\nexists$ continuous $w : B \to \partial B$ s.t. $w|_{\partial B} = id$. Assume there is, and extend w to $\mathbb{R}^n \setminus B$ as $w(x) = x$, $x \notin B$. This is continuous. Also,

$$w(x) \neq 0 \quad \forall x \in \mathbb{R}^n \quad \left(\text{Since } \begin{array}{l} w(x) = x,\ x \notin B, \\ |w(x)| = 1,\ x \in B \end{array}\right)$$

Fix $\varepsilon > 0$ so small that

$$w_\varepsilon = w * \eta_\varepsilon$$

satisfies $w_\varepsilon(x) \neq 0 \quad \forall x \in \mathbb{R}^n$; η_ε radial $\Rightarrow$

$$w_\varepsilon(x) = x \text{ if } x \in \mathbb{R}^n \setminus B(0,2), \quad \varepsilon > 0 \text{ small enough.}$$

[$|x| > 2$, $|\varepsilon| < 1$

$$w_\varepsilon(x) = \int (x-y)\,\eta_\varepsilon(y)\,dy$$

$$= x \underbrace{\int \eta_\varepsilon(y)\,dy}_{=1} - \underbrace{\int y\,\eta_\varepsilon(y)\,dy}_{=0 \text{ odd under reflection!}} = x. \quad]$$

Let

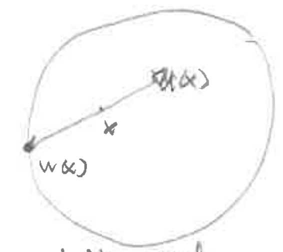
$$\tilde{w} = \frac{2 w_\varepsilon}{|w_\varepsilon|},$$

Then $\tilde{w} : \overline{B(0,2)} \to \partial B(0,2)$ is smooth and by scaling we find a smooth continuous map $B \to \partial B$. ↯.

<u>Step III</u> Finally, Assume $u : B \to B$ continuous, $u(x) \neq x \ \forall x \in B$. Let

$$w : B \to \partial B, \quad w(x) = \text{intersection of } \partial B \text{ and ray from } u(x) \text{ to } x.$$

Then w continuous ↯. □



3.C. <u>Existence of minimizer: coercivity and lower semi-continuity.</u>

Consider

$$I[w] := \int_U L(Du(x), u(x), x)\,dx,$$

$$w|_{\partial U} = g$$

<u>Trivial obs:</u> $\exists$ smooth function bnd from below does not need to attain its minimum: $f = e^{\pm x}$, $= \frac{1}{1+x^2}$

To avoid this we assume the following <u>coercivity</u> <u>condition</u>:

- $1 < q < \infty$, U bnd,
- $\exists$ Constants $\alpha > 0$, $\beta \geq 0$ s.t.

$$L(p,z,x) \geq \alpha |p|^q - \beta \quad \forall\ p \in \mathbb{R}^n, z \in \mathbb{R}, x \in U.$$

Hence

$$I[w] \geq \int \alpha |Dw|^q - \beta \, dx \geq \underbrace{\alpha \|Dw\|^q_{L^q(U)} - \delta}_{\to \infty}, \quad \delta = \beta |U|.$$

as $\|Dw\|_{L^q} \to \infty$.

<u>Note</u>: For $L(p) = \langle p, A(x) p \rangle + f(x)$, A coercive, we can take $q = 2$.

For $L(p) = (1 + |p|^2)^{1/2}$ we'd need $q = 1$, which is beyond us at the moment.

We define (<u>for L coercive</u>)

<u>Def.</u> $\mathcal{A} := \mathcal{A}_g := \{ w \in W^{1,q}(U);\ Tr(w) = g \}$

"The set of admissible functions".

We also allow $I[w]$ to take value $+\infty$!

Assume that

$$m = \inf_{w \in \mathcal{A}_g} I[w].$$

When is the infimum actually attained?

Choose $u_k \in \mathcal{A}$ s.t.

$$\lim_k I[u_k] = m.$$

$\{u_k\}$ is a <u>minimizing</u> <u>sequence</u>.

Idea is to show $\{u_k\}$ converges to an actual minimizer.

Since

$$\begin{cases} \lim_k I[u_k] = m \\ I[u_k] \geq \alpha \|Du_k\|^q_{L^q(U)} - \delta \end{cases}$$

$\Rightarrow$ $\{Du_k\}$ is a bnd sequence in $L^q(U)$. This does not guarantee that $\exists$ conv. subsequence. (No compactness!) However, bnd sets of L^q are weakly seq. compact (Banach-Alaoglu) if $1 < q < \infty$. Hence we def.

<u>Def.</u> $u_{k_j} \to u$ <u>weakly</u> in $W^{1,q}(U)$

($u_{k_j}, u \in W^{1,q}(U)$) if

$$\begin{cases} u_{k_j} \xrightarrow{w} u \text{ in } L^q(U) \\ Du_{k_j} \xrightarrow{w} Du \text{ in } L^q(U). \end{cases}$$

Denote this as $u_{k_j} \xrightarrow{w} u$ in $W^{1,q}(U)$.

So, if g is smooth enough, a minimizing sequence $\{u_k\}$ has a weakly convergent subsequence,

$$u_{k_j} \xrightarrow{w} u \in L^q(U).$$

However, it may happen that this is not enough to guarantee

$$I[u_{k_j}] \longrightarrow I[u] \, !$$

It is actually enough to have:

Def. Functional I is (sequentially) weakly lower semicontinuous if

$$u_k \xrightarrow{w} u \text{ in } W^{1,q}(U) \implies$$

$$I[u] \leq \liminf_k I[u_k].$$

3.7. Convexity

Recall that we showed that at a (smooth minimizer u we had (second variation)

$$\sum_{i,j} \frac{\partial^2 L}{\partial p_i \partial p_j}(Du, u, x)\, \xi_i \xi_j \geq 0,$$

i.e. $p \mapsto L(p, z, x)$ is convex at u. We can actually prove the important result:

Thm. Assume L is smooth, bnd from below and

$$p \mapsto L(p, z, x)$$

is convex $\forall z, x$. Then $I[\cdot]$ is weakly lower semi-cont.

Pf. Step I

Let $u_k \xrightarrow{w} u$ in $W^{1,q}(U)$. We must show

$$I[u] \leq \ell, \quad \ell := \liminf_k I[u_k]$$

Since

$$(u_k, f) \underset{k}{\longmapsto} (u, f) \quad \forall f \in L^p(U), \quad p^* = q$$

$\Rightarrow$

$$|(u_k, f)| \leq C + \|u\|_{L^q(U)} \quad \forall \|f\|_p = 1$$

$\Rightarrow$ (u_k) a bounded sequence in $L^q(U)$. Similarly, (Du_k) is bnd. in $L^q(U)$ $\therefore$ (u_k) is a bnd. seq. in $W^{1,q}(U)$.

By passing to a subsequence, we may also assume

$$\ell = \lim_k I[u_k].$$

Since $W^{1,q}(U) \longrightarrow L^q(U)$ cpt, by passing to yet another subseq. we may assume ($u_k \to u$ in $L^q(U)$!)

$$u_k \longrightarrow u \quad \text{a.e. in } U.$$

Egoroff's Thm $\Rightarrow$ given $\varepsilon > 0$ $\exists$ set $E_\varepsilon \subset U$ s.t.

$$\begin{cases} u_k \longrightarrow u \quad \text{uniformly in } E_\varepsilon \\ \text{Measure of the exceptional set } U \setminus E_\varepsilon \text{ is } < \varepsilon. \end{cases}$$

Assume also

$$E_{\varepsilon'} \subset E_\varepsilon \quad \forall \quad 0 < \varepsilon' < \varepsilon .$$

Let

$$F_\varepsilon := \left\{ x \in U \,;\, |u(x)| + |Du(x)| \leq \frac{1}{\varepsilon} \right\}$$

Since $u \in W^{1,q}(U)$,

$$\lim_{\varepsilon \to +0} m(U \setminus F_\varepsilon) = 0 ,$$

Set now

$$G_\varepsilon := E_\varepsilon \cap F_\varepsilon .$$

Notice that

$$E_\varepsilon = G_\varepsilon \overset{\text{disjoint}}{\cup} (E_\varepsilon \cap (U \setminus F_\varepsilon))$$

$\Rightarrow$

$$m(E_\varepsilon) = m(G_\varepsilon) + m(E_\varepsilon \cap (U \setminus F_\varepsilon)) \leq m(G_\varepsilon) + m(U \setminus F_\varepsilon),$$

and hence

$$m(G_\varepsilon) \geq m(E_\varepsilon) - m(U \setminus F_\varepsilon) > 0 \quad \text{when } \varepsilon \text{ small enough.}$$

and

$$U \setminus G_\varepsilon = (U \setminus E_\varepsilon) \cup (U \setminus F_\varepsilon)$$

$$\Rightarrow \quad m(U \setminus G_\varepsilon) \longrightarrow 0 \quad \text{as} \quad \varepsilon \to +0 .$$

Step II

Assume now (for a moment) $\beta = 0$, i.e.

$$L \geq 0 .$$

Then

$$I[u_k] = \int_U L(Du_k, u_k, x)\,dx \overset{L\geq 0}{\geq} \int_{G_\varepsilon} L(Du_k, u_k, x)\,dx$$

$$= \int_{G_\varepsilon} L(Du + D(u_k - u), u_k, x)\,dx$$

$$= \int_{G_\varepsilon} L(Du, u_k, x)\,dx + \int_{G_\varepsilon} \frac{\partial L}{\partial p}(Du, u_k, x)(Du_k - Du)\,dx$$

$$+ \underbrace{\frac{1}{2}\int_0^1 \int_{G_\varepsilon} \frac{\partial^2 L}{\partial p}\big((1-s)Du + s(Du_k - Du), u_k, x\big)}_{> 0}\,dx\,ds \cdot \langle (Du_k - Du), Du_k - Du \rangle\,dx$$

Linearisation
+ estim. of remainder using convexity

Taylor's remainder term!

$$\geq \int_{G_\varepsilon} L(Du, u_k, x)\,dx + \int_{G_\varepsilon} \frac{\partial L}{\partial p}(Du, u_k, x)(Du_k - Du)\,dx$$

Now

$$\int_{G_\varepsilon} L(Du, u_k, x)\,dx \xrightarrow[k \to \infty]{\text{D.Conv}} \int_{G_\varepsilon} L(Du, u, x)\,dx .$$

Also,

$$\frac{\partial L}{\partial p}(Du, u_k, x) \longrightarrow \frac{\partial L}{\partial p}(Du, u, x) \quad \text{uniformly in } G_\varepsilon$$

and

$$Du_k \xrightarrow{w} Du \quad \text{in } L^q(U),$$

Hence,

$$\int_{G_\varepsilon} \frac{\partial L}{\partial p}(Du, u_k, x)\cdot(Du_k - Du)\,dx \longrightarrow 0.$$

$$\left[\begin{array}{l} f_k \to f \text{ unif. in } G_\varepsilon \\ g_k \xrightarrow{w} g \text{ in } G_\varepsilon \\ g_k \text{ bnd.} \end{array}\right. \qquad \int_{G_\varepsilon} f_k g_k = \int_{G_\varepsilon} \underbrace{(f_k - f)}_{\to 0 \text{ unif.}} \underbrace{g_k}_{\text{bnd}} + \underbrace{\int_{G_\varepsilon} f g_k}_{\to 0}$$

Hence:

$$\liminf_k I[u_k] \geq \int_{G_\varepsilon} L(Du, u, x)\,dx \qquad \forall\, \varepsilon > 0$$

Since $L \geq 0$, Monot. Conv. Thm $\Rightarrow$

$$\liminf_k I[u_k] \geq \int_U L(Du, u, x)\,dx = I[u]. \;\square$$

<u>Thm.</u> (Uniqueness) Under the above assumptions, the minimizer is unique, <u>if L is independent of z.</u>

Pf. Assume $u, \tilde{u} \in \mathcal{A}$ are both minimizers.

Let $v = \dfrac{u + \tilde{u}}{2} \in \mathcal{A}$. We claim

$$I[v] \leq \frac{I[u] + I[\tilde{u}]}{2}$$

<u>and</u> if $\{u = \tilde{u}\}$ is not $= U$ a.e. then $I[v] < \dfrac{I[u] + I[\tilde{u}]}{2}$.

This will be a contradiction.

Unif. Conv. $\Rightarrow$

$$L(p, x) \geq L(q, x) + \frac{\partial L}{\partial p}(q, x)\cdot(p - q) + \frac{\theta}{2}|p - q|^2, \quad \theta \geq 0$$

Take

$$q = \frac{Du + D\hat{u}}{2}, \quad p = Du:$$

$$L(Du, x) \geq L(Dv, x) + \frac{1}{2}\frac{\partial L}{\partial p}(Du, x)(Du - D\hat{u}) + \frac{\theta}{8}|Du - D\hat{u}|^2$$

$\Rightarrow$ int. over U

$$I[u] \geq I[v] + \frac{1}{2}\int \frac{\partial L}{\partial p}(Dv, x)(Du - D\hat{u})\,dx + \frac{\theta}{8}\int_U |Du - D\hat{u}|^2$$

Then take $p = D\hat{u}$, $q = \dfrac{Du + D\hat{u}}{2}$:

$$I[\hat{u}] \geq I[v] + \frac{1}{2}\int \frac{\partial L}{\partial p}(Dv, x)(D\hat{u} - Du)\,dx + \frac{\theta}{8}\int_U |Du - D\hat{u}|^2$$

$\therefore$ adding

$$\frac{I[\hat{u}] + I[u]}{2} \geq I[v] + \frac{\theta}{8}\int_U |Du - D\hat{u}|^2\,dx$$

Also, if

$$I[v] = \frac{I[u] + I[\hat{u}]}{2},$$

then $\int_U |Du - D\hat{u}|^2\,dx = 0 \Rightarrow u - \hat{u} = \text{Const.}$

$u|_{\partial U} = \hat{u}|_{\partial U} \Rightarrow u = \hat{u}.\;\square$

3.8. <u>Weak Solutions of Euler-Lagrange eqns</u>

Note that the introductory computations $\Rightarrow$

<u>u smooth</u> minimizer $\Rightarrow$ u solves (weak) E-L.

To control u we make the following assumptions:

$$(\because)\quad \begin{cases} |L(p,z,x)| \le C(|p|^q + |z|^q + 1) \\ \begin{cases} |\partial L/\partial p(p,z,x)| \le C(|p|^{q-1} + |z|^{q-1} + 1) \\ |\partial L/\partial z(p,z,x)| \le C(|p|^{q-1} + |z|^{q-1} + 1) \end{cases} \quad \forall\, p,z,x. \end{cases}$$

Def. $u \in \mathcal{A}$ is a weak solution of Euler-Lagrange eqn corresponding to L if

$$(*)\quad \int_U \sum_i \frac{\partial L}{\partial p_i}(Du,u,x)\frac{\partial v}{\partial x_i} + \frac{\partial L}{\partial z}(Du,u,x)\, v\, dx = 0 \quad \forall\, v \in W_0^{1,q}(U).$$

Remark.

$$\frac{1}{q'} = 1 - \frac{1}{q} = \frac{q-1}{q}$$

$$\left|\frac{\partial L}{\partial p_i}(Du,u,x)\right| \le C(|Du|^{q-1} + |u|^{q-1} + 1);$$

Now

$$\left(\frac{1}{q}+\frac{1}{q'}=1\right)^{(x)} \int |Du|^{(q-1)q'}dx = \int |Du|^q dx < \infty \quad \text{etc. for others}$$

$$\Rightarrow \frac{\partial L}{\partial p_i}(Du,u,x),\ \frac{\partial L}{\partial z}(Du,u,x) \in L^{q'}(U)$$

and hence $(*)$ makes sense!

Thm. Assume $(\because)$ holds and $u \in \mathcal{A}$ satisfies

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

Then u satisfies $(*)$.

Let

$$i(\tau) = I[u+\tau v], \quad v \in W_0^{1,q}(U).$$

We can't immediately differentiate under integral sign to get $(*)$. However,

$$\frac{i(\tau)-i(0)}{\tau} = \int_U \frac{L(Du+\tau Dv, u+\tau v, x) - L(Du,u,x)}{\tau}\,dx$$

$$=: \int_U L^\tau(x)\,dx$$

$$L^\tau(x) \longrightarrow \sum \frac{\partial L}{\partial p_i}L(Du,u,x)\frac{\partial v}{\partial x_i} + \frac{\partial L}{\partial z}L(Du,u,x)\, v \quad \text{a.e.}$$

Also, Taylor Remainder

$$L^\tau(x) = \frac{1}{\tau}\int_0^\tau \frac{d}{ds}L(Du+sDv, u+sv, x)\,dx$$

$$= \frac{1}{\tau}\int_0^\tau \sum_i \frac{\partial L}{\partial p_i}(Du+sDv, u+sv, x)\frac{\partial v}{\partial x_i} + \frac{\partial L}{\partial z}(Du+sDv, u+sv, x)\, v\, ds$$

Young's ineq.

$$ab \le \frac{a^q}{q} + \frac{b^{q'}}{q'}$$

$\Rightarrow$ (use $(\because)$)

$$|L^\tau| \le C(|Du|^q + |u|^q + |Dv|^q + |v|^q + 1)$$

$\therefore$ Dom. Conv. Thm we may take limit $\tau \to 0$ inside to conclude the claim. $\square$