$\therefore$

$$\limsup_{j,\ell} \|u_{m_j} - u_{m_\ell}\| \le \underbrace{\limsup_{j,\ell} \|u^\varepsilon_{m_j} - u_{m_j}\|}_{\le \varepsilon/2} + \underbrace{\limsup_{j,\ell} \|u^\varepsilon_{m_\ell} - u_{m_\ell}\|}_{\le \varepsilon/2}$$
$$+ \underbrace{\limsup_{j,\ell} \|u^\varepsilon_{m_j} - u^\varepsilon_{m_\ell}\|}_{=0}$$
$$\le \delta .$$

<u>Step VII</u>: Apply previous with $\delta = \delta_n = 1/n$, $n = 1, 2, \dots$) to choose via Cantor-diagonal argument a subsequence $\{u_{m_\ell}\}$ that satisfies

$$\limsup_{k,\ell} \|u_{m_\ell} - u_{m_k}\|_{L^q(V)} = 0 . \quad \square$$

<u>Remark</u>: · As we've seen $W^{1,p}(U) \subset L^p(U)$ cpt $\forall\, 1 \le p \le \infty$
(∂U is C^1) (Ascoli-Arzelà for $p > n$!)

· Always $W^{1,p}_0(U) \subset L^p(U)$ is cpt if U bnd!

1.11. <u>Characterize Fourier-transform</u> ($p=2$)

Consider now $p = 2$, and take $U = \mathbb{R}^n$. Then

$$F : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$$

is a bnd. isomorphism (isometry with correct def. of F)

and ($f \in C^\infty_0(\mathbb{R}^n)$)

$$\widehat{\partial_j f}(\xi) = \partial_j \int e^{i\langle x,\xi\rangle} f(x)\,dx = i\xi_j \int e^{i\langle x,\xi\rangle} f(x)\,dx$$
$$= i\xi_j \hat{f}(\xi) .$$

Gen.

$$\widehat{\partial^\alpha_x f}(\xi) = (i\xi)^\alpha \hat{f}(\xi) .$$

Hence

$$f \in H^k(\mathbb{R}^n) \Leftrightarrow \xi^\alpha \hat{f} \in L^2(\mathbb{R}^n) \;\forall\, |\alpha| \le k$$
$$\Leftrightarrow (1+|\xi|^2)^{k/2} \hat{f} \in L^2(\mathbb{R}^n) \;\forall\, |\alpha| \le k$$

and an equiv. norm on $H^k(\mathbb{R}^n)$ is def. by

$$\|f\|_{H^k} = \left(\int (1+|\xi|^2)^k |\hat{f}(\xi)|^2 d\xi \right)^{1/2} .$$

Generally we can define

$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}, \quad s \ge 0 .$$

~~[illegible]~~.

For any $s \in \mathbb{R}$ we define