I FUNKTIOAVARUUKSISTA

1.1. Hölder-avaruudet

Ol. $U \subset \mathbb{R}^n$ avoin, $0 < \gamma \leq 1$. Palautetaan mieliin, että funktio $u : \bar{U} \to \mathbb{C}$ (tai $\bar{U} \to \mathbb{R}$) on <u>Lipschitz-jva</u> ($\bar{U}$:ssa), jos $\exists$ vakio $C = C_{u,U}$ s.e.

(1.1.1) $\forall x, y \in \bar{U}: \quad |u(x) - u(y)| \leq C|x-y|.$

Huomaa, että i) Lip-jva $\Rightarrow$ jva.

ii) $u \in C^1(U) = \{ f : U \to \mathbb{C},\ f, \frac{\partial f}{\partial x_i} \text{ jvia } \forall i \}$

$\Rightarrow$ u Lip-jva jokaisessa $\tilde{U} \subset U$ s.e. $\bar{\tilde{U}} \subset U$ ja $\bar{\tilde{U}}$ kompakti (HT; käytä väliarvolausetta)

<u>Määr. 1.1.1.</u> Funktio $u : \bar{U} \to \mathbb{C}$ on Hölder-jva eksponentilla γ jos $\exists$ vakio $C = C_{u,U}$ s.e.

$$\forall x, y \in \bar{U}: \quad |u(x) - u(y)| \leq C|x-y|^\gamma.$$

<u>Esim. 1.1.2.</u> $u : [-1,1] \to \mathbb{R}$, $x \mapsto \sqrt{x}$ ei ole Lip-jva, mutta on Hölder-jva eksponentilla $\gamma = \frac{1}{2}$

(HT)

<u>Määr. 1.1.3.</u>

(a) Jos $u : \bar{U} \to \mathbb{C}$ rajoitettu & jva, olkoon

$$\|u\|_{C(\bar{U})} := \sup_{x \in \bar{U}} |u(x)|$$

(b) $$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{\substack{x,y \in \bar{U} \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x-y|^\gamma} \right\}$$

ja määritellään Hölder γ-normi asettamalla

$$\|u\|_{C^{0,\gamma}(\bar{U})} = \|u\|_{C(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}$$

<u>HT</u>: i) Onko $[\cdot]_{C^{0,\gamma}(\bar{U})}$ <u>normi</u>?

ii) os. että $\|\cdot\|_{C(\bar{U})}$ ja $\|\cdot\|_{C^{0,\gamma}(\bar{U})}$ määräävät normit (silloin kun ovat yleensä äärellisiä)

<u>Määr. 1.1.4</u> Olkoon $k \in \{0,1,2,\dots\}$, ja

$$u \in C^k(\bar{U}) = \{ u, \partial^\alpha u / \partial x^\alpha \in C(\bar{U})\ \forall\ |\alpha| \leq k \}.$$

Hölder-avaruus $C^{k,\gamma}(U)$ on

$$C^{k,\gamma}(U) = \{ u \in C^k(\bar{U});\ \|u\|_{C^{k,\gamma}(\bar{U})} \leq \infty \},$$

missä

$$\|u\|_{C^{k,\gamma}(\bar{U})} = \sum_{|\alpha| \leq k} \left\| \frac{\partial^\alpha u}{\partial x^\alpha} \right\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})}.$$

Hölder-avaruudet ovat aina Banach-avaruuksia:

Lause 1.1.5. $C^{k,\gamma}(U)$, $k \in \{0,1,\dots\}$, $0 \le \gamma < 1$, on Banach-avaruus normin $\|\cdot\|_{C^{k,\gamma}(U)}$ suhteen.

Tod. HT. Vihje: kannattaa keskittyä tapaukseen $k=0$, ja siihen että $C^{0,\gamma}(U)$ on täydellinen normin $\|\cdot\|_{C^{0,\gamma}(\bar U)}$ suhteen.
Vihje: Jos (u_j) on Cauchy-jono normin $\|\cdot\|_{C^{0,\gamma}(\bar U)}$ suhteen, on se myös Cauchy-jono avaruudessa $C(\bar U)$. Riittää siis os. että raja-arvo $u(x) = \lim_j u_j(x)$ on Hölder-jvn eksponentilla γ.

Hölder-avaruudet ovat helppoja ymmärtää, ja niillä on monia miellyttäviä ominaisuuksia. Esim. jos $u, v \in C^{0,\gamma}(U)$, niin $uv \in C^{0,\gamma}(U)$.

Tod. $\forall x,y \in U$:

$$|x-y|^{-\gamma}\,|u(x)v(x)-u(y)v(y)| \le \{|u(x)||v(x)-v(y)| + |v(y)||u(x)-u(y)|\}|x-y|^{-\gamma}$$
$$\le \|u\|_{C(\bar U)}[v]_{C^{0,\gamma}(\bar U)} + \|v\|_{C(\bar U)}[u]_{C^{0,\gamma}(\bar U)}$$

□

Valitettavasti Hölder-ratkaisujen konstruointi ei ole olennaisesti helpompaa kuin klassisten ratkaisujen, päinvastoin.

1.2. Heikot derivaatat

Sen sijaan että pyrkisimme kuvailemaan mahdollisten ratkaisujen "singulariteetteja" tarkasti Hölder-avaruuksien avulla, lievennämme derivoituvuuden käsitettä:

Esim. 1.2.1. a) Olkoon $f, \phi \in C^1([-1,1])$. Tällöin os. integroinnilla

$$\int_{-1}^{1} f'\phi\,dx = \Big/_{-1}^{1} f\phi - \int_{-1}^{1} f\phi'\,dx.$$

Jos nyt $\phi(\pm 1) = 0$, niin pätee

$$\int_{-1}^{1} f'\phi\,dx = -\int_{-1}^{1} f\phi'\,dx.$$

Otamme tämän triviaalin havainnon f':n heikon määritelmän lähtökohdaksi: Funktio $g \in L^1(-1,1)$ (huom: vain integroituvuus!) on $f \in L^1(-1,1)$:n heikko derivaatta välillä $(-1,1)$ jos $\forall \phi \in C^1([-1,1])$, $\phi(\pm 1) = 0$, pätee

$$\int_{[-1,1]} g\phi\,dx = -\int_{[-1,1]} f\phi'\,dx$$ (Jos niin $f \in C^1([-1,1])$ valmme aina valita $g = f'$.)

b) Olkoon $f(x) = |x|$, $|x| \le 1$. Nyt $f \in C([-1,1]) \subset L^1([-1,1])$, mutta f ei ole derivoituva origossa. Onko olemassa heikko derivaatta f'?

Vast: On: olkoon $g(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$.

Tällöin $\forall \phi \in C^1([-1,1])$, $\phi(\pm 1) = 0$, pätee

$$\int_{[-1,1]} f\phi'\,dx = \int_{-1}^{0} |x|\cdot\phi'(x)\,dx + \int_{0}^{1} |x|\cdot\phi'(x)\,dx = \int_{-1}^{0} -x\phi'\,dx + \int_{0}^{1} x\phi'\,dx$$

$$\overset{\text{os. int}}{=} \underbrace{\Big/_{-1}^{0} x\phi}_{=0} + \int_{-1}^{0}\phi\,dx + \underbrace{\Big/_{0}^{1} x\phi}_{=0} - \int_{0}^{1}\phi\,dx = -\int_{-1}^{1} g\phi\,dx.$$

c) Olkoon $h(x) = \chi_{[0,1]} = \begin{cases} 0, & -1 < x \le 0 \\ 1, & 0 \le x < 1 \end{cases}$. Onko h:lla heikkoa derivaattaa?

<u>Raht.</u> Jos $g \in L^1[-1,1]$ on h:n heikko derivaatta, pätee siis

$$\int_{[-1,1]} g\phi\,dx = -\int_{[-1,1]} h\phi'\,dx \qquad \forall\, \phi \in C^1([-1,1]),\ \phi(\pm 1) = 0.$$

Nyt

$$\int_{[-1,1]} h\phi'\,dx = \int_{[-1,0]} \underbrace{h}_{h\equiv 0}\phi'\,dx + \int_{[0,1]} \underbrace{h}_{h\equiv 1}\phi'\,dx = \int_0^1 \phi'\,dx = \phi(1) - \phi(0).$$

Siis g:n on toteutettava

$$\int_{[-1,1]} g\phi\,dx = \phi(0) \quad \forall\, \phi \in C^1([-1,1]),\ \phi(\pm 1) = 0.$$

Jos lis. $\phi(0) = 0$, niin

$$\int_{[-1,1]} g\phi\,dx = 0.$$

Tällöin (mieti miksi!) $g = 0$ m.k. kun $0 < |x| < 1$ ja siis ($g \in L^1[-1,1]$) $\Rightarrow g = 0$ m.k.

$$\Rightarrow \int_{[-1,1]} g\phi\,dx = 0 \ \forall\, \phi,\ \phi \in C^1([-1,1]),\ \phi(\pm 1) = 0.\ \lightning.$$

h:lla ei ole heikkoa L^1-derivaattaa!

Voimme toimia samoin myös $\mathbb{R}^n$:n avoimissa joukoissa:

Palautetaan mieliin os.derivointi $\mathbb{R}^n$:ssä ja multi-indeksi notaatio:

a) <u>Os.deriv.</u> Jos $\phi, u \in C^1(\bar{U})$, U on raj. C^1-alue, niin

$$\int_U \phi \frac{\partial u}{\partial x_i}\,dx = -\int_U \frac{\partial \phi}{\partial x_i} u\,dx + \int_{\partial U} \nu_i \phi u\,dS$$

dS ∂U:n pintamitta

$\nu = (\nu_1, \dots, \nu_n)$ ∂U:n yks. ulkonormaali

Erityisesti, jos $\phi \in C_0^1(U)$ (eli $\phi \in C^1(\bar{U})$, $\operatorname{supp}\phi \subset U$ kohti)

niin

$$\int_U \phi \frac{\partial u}{\partial x_i}\, dx = -\int_U \frac{\partial \phi}{\partial x_i} u\, dx$$

ja nyt ei tarvitse olettaa U:n reunan sileydestä mitään!

b) Multi-indeksi notaatio Olk. $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_k \in \{0, 1, \dots\}$

Merk.

$$\partial^\alpha u = \frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^{\alpha_1} \cdots \partial^{\alpha_n} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad (u \text{ riitt. sileä})$$

$|\alpha| = \alpha_1 + \dots + \alpha_n$ (α:n pituus)

c) Os. integrointi yleisellä α:

Ol. $\phi \in C_0^{|\alpha|}(U)$, $u \in C^{|\alpha|}(\bar{U})$. Tällöin

$$\int_U \phi\, \partial^\alpha u\, dx = (-1)^{|\alpha|} \int_U \partial_\alpha \phi \cdot u\, dx.$$

Nyt voimme määritellä:

Määr. 1.2.2. Ol. $u, v \in L^1_{loc}(U)$, $U \subseteq \mathbb{R}^n$. Sanomme, että v on u:n heikko α-derivaatta, merk. edelleen $v = \partial_\alpha u = \partial^\alpha u / \partial x^\alpha$, jos

$$\int_U u\, \partial^\alpha \phi\, dx = (-1)^{|\alpha|} \int_U v \phi\, dx \quad \forall \phi \in C_0^\infty(U)$$

Huom. Tämä ehto määrää funktion $v \in L^1_{loc}(U)$ yksikäsitteisesti (HT!)

Tärkeää

Vielä yksi huomautus diff.yhtälöistä & heikoista derivaatoista.

Ex. 1.2.3 Let $\chi_{[0,1]} = \begin{cases} 1, & 0<x<1 \\ 0, & \text{else} \end{cases}$

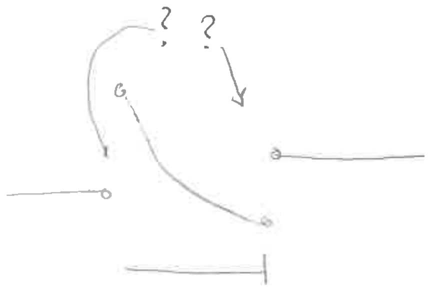
Consider DE $f' + \chi_{[0,1]} f = 0$. (☺)

What do mean by this? Of course

$$\begin{cases} f' = 0, & x<0,\ x>1 \\ f' + f = 0 & 0<x<1 \end{cases}$$

These we know how to solve:

$$f(x) = \begin{cases} f_1(x) = C_1, & x<0 \\ = f_2(x) = C_2, & x>1 \\ = f_3(x) = C_3 e^{-x}, & 0<x<1 \end{cases}$$



What to do at the jumps of cf-function $\chi_{[0,1]}$?

Let's demand that f is a weak sol of (☺) i.e.

$f \in L^1_{loc}$ & $\int f(-\phi' + \chi_{[0,1]}\phi)\,dx = 0 \quad \forall \phi \in C_0^\infty(\mathbb{R})$

Now

$$0 = \int f(-\phi' + \chi_{[0,1]}\phi)dx = -\int_{x<0} f_1\phi'\,dx + \int_{0<x<1} f_3(-\phi'+\phi)\,dx - \int_{x>1} f_2\phi'\,dx$$

$$= \underbrace{\int_{x<0} f_1'\phi\,dx}_{=0} - f_1(0)\phi(0) + \underbrace{\int_{0<x<1}(f_3'+f_3)\phi\,dx}_{=0} - (f_3(1)\phi(1) - f_3(0)\phi(0))$$

$$+ \underbrace{\int_{x>1} f_2'\phi\,dx}_{=0} + f_2(1)\phi(1).$$

This has to hold for all values of $\phi(0)$ & $\phi(1)$

$\Rightarrow$ we must have

$$\begin{cases} f_1(0) = f_3(0) \\ f_2(1) = f_3(1) \end{cases} \iff \begin{cases} C_1 = C_3 e^0 = C_3 \\ C_2 = C_3 e^{-1} \end{cases}$$

$\therefore$ Any weak sol. must be of the form

$$f(x) = \begin{cases} C, & x<0 \\ Ce^{-x}, & 0<x<1 \\ Ce^{-1}, & x>1 \end{cases}, \quad C \in \mathbb{C} \text{ arb.}$$

to fix f uniq. we hence need only 1, say, beh. of f at $\pm\infty$, or at any 1 point.

Motto [Demanding the Diff.-eq. to be valid weakly imposes natural boundary conditions on discontinuities of derivatives]

This characterizes the natural B.C.! in Electromagnetics, formation & location of shock waves etc.....

1.3. Def. of Sobolev spaces

Let $1 \le p \le \infty$, $k \in \{0,1,2,\dots\}$, $U \subseteq \mathbb{R}^n$,

Def. 1.3.1 "Sobolev space $W^{k,p}$ on U"

$$W^{k,p}(U) := \{u \in L^1_{loc}(U);\ \forall |\alpha| \le k, \text{ weak derivatives } \partial^\alpha u \text{ exist and } \in L^p(U)\}$$

<u>Remarks</u> i) From now on we identify functions that agree a.e.

ii) In the special case $p=2$ we denote

$$H^k(U) = W^{k,2}(U)$$

Let

$$\|u\|_{W^{k,p}(U)} := \Big(\sum_{|\alpha|\le k} \|\partial_\alpha u\|^p_{L^p(U)} \Big)^{1/p} \qquad 1\le p<\infty$$

$$\|u\|_{W^{k,\infty}(U)} = \sum_{|\alpha|\le k} \|\partial_\alpha u\|_{L^\infty(U)}$$

<u>Prop. 1.3.2</u> $\|\cdot\|_{W^{k,p}(U)}$ is a norm in $W^{k,p}(U)$.

Pf. Train by the properties of L^p-norms □

<u>Notations</u> As usual, $u_m \xrightarrow{m\to\infty} u$ in $W^{k,p}(U)$ if $u_m, u \in W^{k,p}(U)$ and $\|u_m - u\|_{W^{k,p}(U)} \longrightarrow 0$ as $m\to\infty$.

Also, we say $u_m \to u$ in $W^{k,p}_{loc}(U)$ if $u_m, u \in W^{k,p}(U)$ and

$$\|u_m - u\|_{W^{k,p}(V)} \xrightarrow[m\to\infty]{} 0 \quad \forall\, V \Subset U,\ \bar V \text{ cpt.}$$

The following space will be very important for us:

<u>Def. 1.3.3.</u> $W^{k,p}_0(U) = \overline{C^\infty_0(U)}^{\,W^{k,p}(U)}$

: closure of $C^\infty_0(U)$ in $W^{k,p}(U)$

i.e. $u \in W^{k,p}_0(U) \iff \exists$ sequence $u_m \in C^\infty_0(U)$ s.t. $\|u_m - u\|_{W^{k,p}(U)} \to 0$.

<u>Note</u>: It is not that easy to understand when a given $u \in W^{k,p}(U)$ belongs to $W^{k,p}_0(U)$.
However, for us $W^{k,p}_0(U)$ is the subspace of those functions in $W^{k,p}(U)$ whose boundary values of ∂U vanish to order $k-1$ i.e.

$$\text{“} \partial^\alpha u = 0 \text{ on } \partial U, \ |\alpha| \le k-1 \text{”}.$$

More about this later.

Also, we write $H^k_0(U) = W^{k,2}_0(U)$.

Let's look at two examples:

<u>Ex. 1.3.4.</u> $U = \{x \in \mathbb{R}^n;\ |x|<1\}$,

$u(x) = |x|^{-\alpha}$, $x \neq 0$.

when [illegible] $W^{1,p}(U)$?

Now $1 \le p < \infty$,

$$\|u\|^p_{L^p(U)} = \int_{|x|\le 1} |x|^{-\alpha p}\, dx = \int_0^1 \int_{|x|=1} r^{-\alpha p} r^{n-1}\, dr\, d\omega \qquad \omega_n = \text{area of } \{|x|=1\} \text{ in } \mathbb{R}^n.$$

$$= \omega_n \int_0^1 r^{n-\alpha p-1}\, dr < \infty$$

iff $n - \alpha p > 0$.

If $p=\infty$, then $u \in L^\infty(U)$ iff $\alpha \le 0$.

What about derivatives?

$$\partial_{x_i} u = -\alpha |x|^{-\alpha-1} \cdot \partial_{x_i}|x| = -\alpha |x|^{-\alpha-2} x_i \, , \quad x \neq 0 .$$

Note that we <u>also</u> need to show that $\exists$ weak deriv. in U!

Let $\phi \in C_0^\infty(U)$; fix $\varepsilon > 0$.

$$\int_{\varepsilon<|x|<1} u \, \partial_{x_i}\phi \, dx \overset{\text{Green}}{=} -\int_{|x|=\varepsilon} \nu_i u \phi \, dS - \int_{\varepsilon<|x|<1} \partial u_{x_i} \phi \, dx$$

$\phi|_{|x|=1} = 0$!

$\nu = (\nu_1, \dots, \nu_n)$ is the exterior unit normal of $\{|x| < \varepsilon\}$.
$= |x|/\varepsilon$.

Then

$$\int_{|x|=\varepsilon} \nu_i u \phi \, dS = \int_{|x|=\varepsilon} \nu_i \varepsilon^{-\alpha} \phi \, dS \xrightarrow[\text{as } \varepsilon \to 0]{\forall \phi} 0 \quad \text{iff} \quad n-1-\alpha > 0$$

∂ has area $\sim \varepsilon^{n-1}$

$\iff$ $\underline{n > \alpha + 1}$

Then

$$\int_U u \, \partial_{x_i}\phi \, dx = -\int_U \partial_{x_i} u \, \phi \, dx \quad \forall \phi \in C_0^\infty(U),$$

and also $\partial_{x_i} u \in L^1_{loc}(U)$ $\quad -\alpha - 1 > -n \iff n > 1 + \alpha$.

So story so far:

$$u \in L^p \text{ iff } \begin{cases} n - \alpha p > 0, & 1 \le p < \infty \\ \alpha < 0, & p = \infty \end{cases}$$

u has weak derivatives iff $n > \alpha + 1$.

So, when $\partial_{x_i} u \in L^p(U)$?

If $1 \le p < \infty$,

$$\|\partial_{x_i} u\|^p_{L^p(U)} = \int_{|x|<1} \alpha^p (|x_i| |x|^{-\alpha-2})^p \, dx$$

$$\le \alpha^p \int_{|x|<1} |x|^{(-\alpha-1)p} \, dx < \infty \quad \text{iff} \quad -(\alpha+1)p > -n$$

$\iff n > (\alpha+1)p \ge \alpha + 1$ since $p \ge 1$

If $p = \infty$, then $\partial_{x_i} u \in L^\infty(U)$ iff $-\alpha - 1 \ge 0 \iff \underline{\alpha \le -1}$

Let's collect these:

$u \in W^{1,p}(U)$, $1 \le p < \infty$ iff $n > (\alpha+1)p$

$\iff \alpha < \frac{n-p}{p}$

$u \in W^{1,\infty}(U)$ iff $\alpha \le -1$.

<u>Ex. 1.3.5</u> Let $\{r_k\}_{k=1}^\infty$ be a countable, dense subset of $\{x \in \mathbb{R}^n ; |x| < 1\}$.

We can turn the previous, natural, example to more bizarre by "condensating singularities":

Let $u(x) = \sum_{k=1}^\infty 2^{-k} |x - r_k|^{-\alpha}$.

Then $u \in W^{1,p}(U)$ iff $(\alpha+1)p < n$ but u is unbounded on all open subsets of U. (HW: Think why $\exists$ uniform bound on $W^{1,p}(U)$ norms of $|x - r_k|^{-\alpha}$)

Hence $W^{1,p}$ does not guarantee nice behaviour. However we'll see later that $p > n \Rightarrow W^{1,p} \subset C^{0,\gamma}$ for some $\gamma > 0$.

1.4. Elementary properties

First some elementary properties of weak derivatives:

Prop. 1.4.1. Assume $u, v \in W^{k,p}(U)$ and $|\alpha| \leq k$.

Then $\qquad\qquad (D = \partial/\partial x)$

(i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$

$$D^{\alpha+\beta} u = D^\alpha(D^\beta u) = D^\beta(D^\alpha u) \ \forall \ \alpha, \beta, |\alpha|+|\beta| \leq k.$$

(ii) $\forall \ \lambda, \mu \in \mathbb{C}$. $\lambda u + \mu v \in W^{k,p}(U)$ and

$$D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v \quad, |\alpha| \leq k.$$

(iii) $V \Subset U \Rightarrow u|_V \in W^{k,p}(V)$.

(iv) If $\zeta \in C_0^\infty(U)$, then $\zeta u \in W^{k,p}(U)$

and (Leibniz)

$$D^\alpha(\zeta u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^\beta \zeta \, D^{\alpha-\beta} u$$

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!} = \frac{\alpha_1! \cdots \alpha_n!}{\beta_1! \cdots \beta_n! (\alpha_1-\beta_1)! \cdots (\alpha_n-\beta_n)!}$$

Pf. Note that since we are dealing with weak derivatives one should not treat these as obvious!

(i) Let $|\alpha| + |\beta| \leq k$. Then $\forall \ \phi \in C_0^\infty(U)$:

$$\int_U D^\alpha u \underbrace{D^\beta \phi}_{\in C_0^\infty(U)} dx = (-1)^{|\alpha|} \int_U u \, D^\alpha(D^\beta \phi) \, dx = (-1)^{|\alpha|} \int_U u \, D^{\alpha+\beta} \phi \, dx$$

$$= (-1)^{|\alpha|+|\beta|+|\alpha|} \int_U D^{\alpha+\beta} u \, \phi \, dx$$

$$= (-1)^{|\beta|} \int_U \underbrace{D^{\alpha+\beta} u}_{\in L^1_{loc}} \phi \, dx.$$

Hence $D^\alpha u$ has a weak β-derivative, and

$$D^\beta(D^\alpha u) = D^{\alpha+\beta} u.$$

Sim. $D^{\alpha+\beta} u = D^\alpha(D^\beta u)$, and $D^\alpha u \in W^{k-|\alpha|,p}(U)$.

(ii) Trivial

(iii) Trivial, since $C_0^\infty(V) \subset C_0^\infty(U)$ by extension as zero.

(iv) Proof is by induction on α. Assume $|\alpha| = 1$; $\alpha = \partial/\partial x_i$

$$\int \zeta u \, D_{x_i} \phi \, dx = \int u D_{x_i}(\zeta\phi) - u\phi D_{x_i}\zeta \, dx = -\int (\zeta D_{x_i} u + u D_{x_i} \zeta) \phi \, dx.$$

Assume OK for all α, $|\alpha| \leq k-1$. Let $|\alpha| = k \geq 2$ and $\beta \leq \alpha$ s.t. $|\beta| = k-1$, $\beta + \gamma = \alpha$ i.e. $|\gamma| = 1$.

Then

$$\int_U (\zeta u) D^\alpha \phi \, dx = \int_U (\zeta u) D^\beta (D^\gamma \phi) \, dx$$

$$= (-1)^{|\beta|} \int_U D^\beta(\zeta u) D^\gamma \phi \, dx$$

$$\overset{\text{ind ass}}{=} (-1)^{|\beta|} \int_U \sum_{\delta \le \beta} \binom{\beta}{\delta} D^\delta \zeta D^{\beta-\delta} u \, D^\gamma \phi \, dx$$

$$= (-1)^{\underbrace{|\beta|+|\gamma|}_{=|\alpha|}} \int_U \sum_{\delta \le \beta} \binom{\beta}{\delta} D^\gamma \left(D^\delta \zeta D^{\beta-\delta} u \right) \phi \, dx$$

$$\underbrace{\beta+\gamma}-(\delta+\gamma) = \alpha-\delta+\gamma \qquad \underbrace{\gamma+\beta-\delta} = \alpha$$

$$= (-1)^{|\alpha|} \int_U \sum_{\delta \le \beta} \binom{\beta}{\delta} \left(D^{\delta+\gamma} \zeta D^{\beta-\delta} u + D^\delta \zeta D^{\gamma+\beta-\delta} u \right) \phi \, dx$$

$$\overset{\rho=\delta+\gamma}{=} (-1)^{|\alpha|} \int_U \sum_{\delta \le \beta} \binom{\beta}{\delta} \left(D^\rho \zeta D^{\alpha-\rho} u + D^\delta \zeta D^{\alpha-\delta} u \right) \phi \, dx$$

$$= (-1) \int_U \sum_{\delta \le \alpha} \left\{ \underbrace{\binom{\beta}{\delta} + \binom{\beta}{\delta-\gamma}} \right\} D^\delta \zeta D^{\alpha-\delta} \phi \, dx$$

$$\overset{\text{check}}{=} \binom{\alpha}{\delta}$$

cf.

$$\binom{m}{k} + \binom{m}{k-1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k-1)!(m-k+1)!}$$

$$= \frac{m!(m-k+1) + m!\,k}{(m-k+1)!\,k!} = \frac{m!(m-k+1+k)}{(m-k+1)!\,k!} = \frac{(m+1)!}{k!(m-k+1)!} = \binom{m+1}{k}$$

$\square$

$\boxed{\text{Rule of Thumb}}$ One can compute with weak derivatives almost like with ordinary derivatives

The following is important – and easy – since all the hard work was done when proving the completeness of $L^p(U)$:)

<u>Th. 1.4.2.</u> $W^{k,p}(U)$ is Banach.

Pf. Assume (u_m), $u_m \in W^{k,p}(U)$ is Cauchy

$\Leftrightarrow$ $(D^\alpha u_m)$ is Cauchy $\forall\ |\alpha| \le k$.

i.e. $\exists$ $u^\alpha \in L^p(U)$ s.t.

$$D^\alpha u_m \xrightarrow{L^p} u^\alpha.$$

Especially $u_m \xrightarrow{L^p} u := u^0$.

It is enough to prove that $u \in W^{k,p}(U)$ and $D^\alpha u = u^\alpha$.

Now $\forall\ \phi \in C_0^\infty(U)$:

$$\int_U u D^\alpha \phi \, dx = \lim_{m\to\infty} \int_U u_m D^\alpha \phi \, dx = \lim_{m\to\infty} (-1)^{|\alpha|} \int_U (D^\alpha u_m) \phi \, dx$$

$$\overset{\text{since } D^\alpha u_m \xrightarrow{L^p} u^\alpha}{=} (-1)^{|\alpha|} \int u^\alpha \phi \, dx.$$

Hence $D^\alpha u = u^\alpha \in L^p(U)$. $\square$

One can actually prove more if $p=2$.

<u>Cor. 1.4.3</u> $H^k(U)$ is a Hilbert-space.

Pf. It is enough to note that

$$(u,v)_{H^k(U)} := \sum_{|\alpha| \le k} (D^\alpha u, D^\alpha v)_{L^2(U)}$$

defines an inner product in $H^k(U)$ s.t. $(u,u)_{H^k(U)} = \|u\|^2_{H^k(U)}$. $\square$

1.5. Approximation by smooth functions

It will be crucially important for us to be able to approximate a $f \in W^{k,p}(U)$ with $C^\infty(U)$ with much ~~ras~~ control as possible. We will do this in several steps.

However, before going forward, let's recall that C^∞-functions are dense in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$:

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be s.t. $0 \le \phi \le 1$, $\int \phi\, dx = 1$. $\operatorname{supp}\phi \subset \{|x| < 1\}$

Let $\varepsilon > 0$ and define

$$\phi_\varepsilon(x) = \varepsilon^{-n}\phi(x/\varepsilon).$$

Then $\int \phi_\varepsilon\, dx = 1$.

Let $u \in L^p(\mathbb{R}^n)$, and let

$$u_\varepsilon(x) = u * \phi_\varepsilon(x) = \int u(x-y)\,\phi_\varepsilon(y)\,dy = \int u(y)\,\phi_\varepsilon(x-y)\,dy.$$

Dom. Conv. $\Rightarrow$

$$\partial_{x_i} u_\varepsilon(x) = \int u(y)\,\partial_{x_i}\phi_\varepsilon(x-y)\,dy = u * \partial_{x_i}\phi_\varepsilon,$$

and simil. $\forall\ \alpha$:

$$D^\alpha u_\varepsilon(x) = \int u(y)\, D_x^\alpha \phi_\varepsilon(x-y)\,dy = u * D^\alpha \phi_\varepsilon$$

Hence $u_\varepsilon \in C^\infty(\mathbb{R})$.

Now

$$u(x) - u_\varepsilon(x) = u(x) - \int u(x-y)\,\phi_\varepsilon(y)\,dy \overset{\int\phi_\varepsilon dx = 1}{=} \int \big(u(x) - u(x-y)\big)\,\phi_\varepsilon(y)\,dy.$$

Hence

$$|u(x) - u_\varepsilon(x)|^p \underset{\phi_\varepsilon \ge 0}{\le} \left|\int |u(x) - u(x-y)|\,\phi_\varepsilon(y)\,dy\right|^p$$

Let $d\mu_\varepsilon = \phi_\varepsilon\, dy$. Then $\int d\mu_\varepsilon = 1$ $\therefore$ may apply Jensen.

$$\overset{\text{Jensen}}{\le} \int |u(x) - u(x-y)|^p\, d\mu_\varepsilon$$

$$= \int |u(x) - u(x-y)|^p\,\phi_\varepsilon(y)\,dy.$$

$$\overset{y' = y/\varepsilon}{=} \int |u(x) - u(x - \varepsilon y')|^p\,\phi(y')\,dy'$$

and thus

$$\|u - u_\varepsilon\|^p_{L^p(\mathbb{R}^n)} \le \iint |u(x) - u(x - \varepsilon y')|^p\,\phi(y')\,dy'\,dx.$$

Let

$$T_\varepsilon(y') = \int |u(x) - u(x - \varepsilon y')|^p\,dx$$

$$= \|u - u(\cdot - \varepsilon y')\|^p_{L^p(\mathbb{R}^n)}.$$

Translations are continuous in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$

$\Rightarrow$ $T_\varepsilon(y') \to 0$ as $\varepsilon \to +0$.

On the other hand,

$$|T_\varepsilon| \le 2^p \|u\|^p_{L^p(\mathbb{R}^n)}$$

$\therefore$ Dom. conv $\Rightarrow$ $\|u - u_\varepsilon\|^p_{L^p(\mathbb{R}^n)} \to 0$ as $\varepsilon \to +0$.

The 1^{st} result in $W^{k,p}(U)$ - spaces is the following:

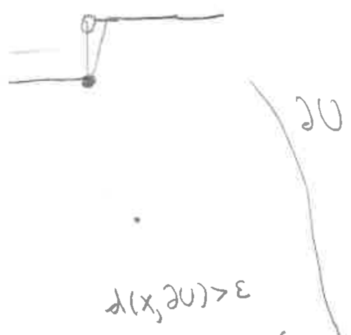
<u>Prop. 1.5.1</u> (Local approximation by C^∞-functions)

Let $u \in W^{k,p}(U)$, $1 \leq p < \infty$.

Given $\varepsilon > 0$, let $\quad | \; U_\varepsilon$ open

$$U_\varepsilon = \{ x \in U ; \, d(x, \partial U) > \varepsilon \}$$

$\eta \in C_0^\infty(B)$, $\int \eta \, dx = 1$, $\eta \geq 0$, $\eta_\varepsilon(x) = \varepsilon^{-n} \eta(x/\varepsilon)$



Then

i) $u_\varepsilon = u * \eta_\varepsilon \in C^\infty(U_\varepsilon)$

ii) If $V \subset\subset U$, V open $\Rightarrow$
[i.e. $\bar{V}$ cpt, $\bar{V} \subset U$]

$$u_\varepsilon \longrightarrow u \text{ in } W^{k,p}(V).$$

$d(x, \partial U) > \varepsilon$
$\Rightarrow d(x, \partial U) > \varepsilon + \delta$ for some δ
$B(x, \delta/2) \subset U_\varepsilon$

[Hence we can approximate u by C^∞-functions if we stay uniformly away from ∂U]

Pf. (i) If $d(x, \partial U) > \varepsilon$, then $x - y \in U \;\forall\; |y| < \varepsilon$,

and hence

$$u_\varepsilon(x) = \int u(x-y) \eta_\varepsilon(y) \, dy = \varepsilon^{-n} \int_{|y| < \varepsilon} u(x-y) \eta(y/\varepsilon) \, dy$$

is well defined and we can diff. under the integral sign to get

$$D^\alpha u_\varepsilon(x) = u * D^\alpha \eta_\varepsilon(x), \quad x \in U_\varepsilon$$

Hence $u_\varepsilon \in C^\infty(U_\varepsilon)$ and $u_\varepsilon \in W^{k,p}(V)$

$\forall$ V open, $V \subset\subset U$, when ε small enough
(so small that $d(\bar{V}, \partial U) > \varepsilon$)

(ii) Now, $\forall x \in U_\varepsilon$

$$D^\alpha u^\varepsilon(x) = \int_U D_x^\alpha \eta_\varepsilon(x-y) u(y) \, dy$$

$$= (-1)^{|\alpha|} \int_U D_y^\alpha \eta_\varepsilon(x-y) u(y) \, dy .$$

With $x \in U_\varepsilon$ fixed, the function

$$y \longmapsto \eta_\varepsilon(x-y)$$

is in $C_0^\infty(U)$ and hence def. of weak deriv. $\Rightarrow$

$$D^\alpha u^\varepsilon(x) = (-1)^{|\alpha|} \int_U D_y^\alpha \eta_\varepsilon(x-y) u(y) \, dy$$

$$= \int_U \eta_\varepsilon(x-y) D^\alpha u(y) \, dy$$

$\therefore$ $D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u$ in U^ε, $D^\alpha u \in L^p(U)$.

Hence - like in the introductory example -

$$D^\alpha u^\varepsilon \xrightarrow{L^p(V)} D^\alpha u$$

Or check Appendix C4 in Evans for full proof!

$$\Rightarrow u_\varepsilon \xrightarrow{W^{k,p}(V)} u . \quad \square$$

Next we push the approximations to whole U:

<u>Prop.</u> 1.5.2. Assume $U \Subset \mathbb{R}^d$ bnd (try to spot if we really use this ∷) $u \in W^{k,p}(U)$, $1 \le p < \infty$. Then $\exists$ seq. (u_m), $u_m \in W^{k,p}(U) \cap C^\infty(U)$ s.t. $u_m \longrightarrow u$ in $W^{k,p}(U)$.

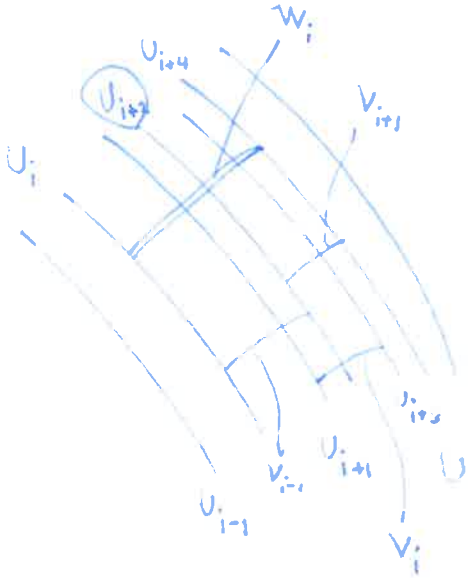
Pf. 1) Let $U_i = \{x \in U;\ d(x, \partial U) > 1/i\}$, $i = 1, 2, \dots$

$$U = \bigcup_{i=1}^\infty U_i$$

Let $V_i = U_{i+3} - \bar{U}_{i+1}$, $i = 1, 2, \dots$

V_0 open s.t. $\bar{V} \subset U$ cpt & $U = \bigcup_{i=1}^\infty V_i$

Choose $0 \le \zeta_i \le 1$ s.t. $\begin{cases} \zeta_i \in C_0^\infty(V_i) \\ \sum_{i=0}^\infty \zeta_i = 1 \end{cases}$



2) Choose $\delta > 0$ arb. Then choose $\varepsilon_i > 0$ s.t.

$u^i := \eta_{\varepsilon_i} * (\zeta_i u)$, (Note: $\operatorname{supp} u^i \subset V_i + \varepsilon_i$

satisfies $\begin{cases} \|u^i - \zeta_i u\|_{W^{k,p}(U)} < \delta\, 2^{-(i+1)} \\ \operatorname{supp} u^i \subset W_i \end{cases}$, $W_i = U_{i+4} - U_i$

3) Let's define - formally - $v = \sum_{i=0}^\infty u^i$.

This is well def & $C^\infty(U)$ since the sum is locally finite. Actually, on a given $K \subset U$ cpt, only fin. many u^i is are $\neq 0$.

Now $u = \sum_{i=0}^\infty \zeta_i u$, we get for any $V \subset U$, $\bar{V} \subset U$ cpt.

$$\|u - v\|_{W^{k,p}(V)} \le \sum_{i=0}^\infty \|u^i - \zeta_i u\|_{W^{k,p}(V)} \le \delta \quad \text{by geom. series.}$$

$$\therefore \quad \|u - v\|_{W^{k,p}(U)} = \sup_{\substack{V \subset U \\ \bar{V} \text{ cpt} \subset U}} \|u - v\|_{W^{k,p}(V)} \le \delta.$$

This holds $\forall\, \delta$

$\Rightarrow \quad u = v$. $\square$

Next step is to push the approximations all the way to ∂U:

Th. 1.5.3. $U \Subset \mathbb{R}^d$ bnd, ∂U is C^1

$u \in W^{k,p}(U)$, $1 \le p < \infty$.

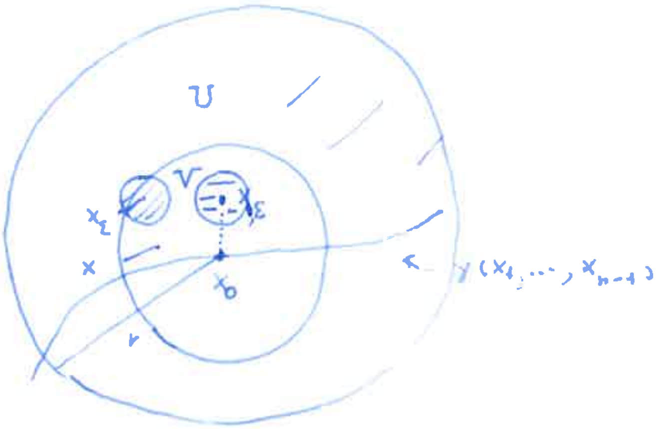
Then $\exists$ sequence (u_m), $u_m \in C^\infty(\bar{U})$ s.t. $u_m \xrightarrow{W^{k,p}(U)} u$.

Pf. 1) Fix $x_0 \in \partial U$ & choose $r > 0$

and $\gamma : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ s.t.

$U \cap B(x_0, r) = \{x \in B(x_0,r);\ x_n > \gamma(x_1, \dots, x_{n-1})\}$

$V = U \cap B(x_0, r/2)$.



2) Let $x \in V$, $\varepsilon > 0$ and

$$x^\varepsilon = x + \lambda \varepsilon e_n$$

Then $B(x^\varepsilon; \varepsilon) \subset U \cap B(x_0, r)$ $\forall\, x \in V$, $\varepsilon > 0$ small enough

$\lambda > 1$ fixed, suff. large.

Define

$$\begin{cases} u_\varepsilon(x) = u(x + \lambda \varepsilon e_n), \quad x \in V. \\ v_\varepsilon(x) = \eta_\varepsilon * \underset{\sim}{u_\varepsilon} \in C^\infty(\bar{V}) \end{cases}$$

has support in $\{x \in U;\ d(x, \partial U) > \varepsilon\}$

Question: why do we need the parameter λ?

Hint: [The boundary looks like a hyperplane + small pert. for ε small enough.

3) (This is where the magic happens!)

Next we prove $v_\varepsilon \longrightarrow u$ in $W^{k,p}(V)$.

Let $|\alpha| \le k$:

$$\|D^\alpha v^\varepsilon - D^\alpha u\|_{L^p(V)} \le \underbrace{\|D^\alpha (v^\varepsilon - u_\varepsilon)\|_{L^p(V)}}_{\to 0} + \underbrace{\|D^\alpha u_\varepsilon - D^\alpha u\|_{L^p(V)}}_{\xrightarrow{\varepsilon \to 0} 0}$$

by the previous mollifier argument

since translations are continuous in L^p, $1 \le p < \infty$.

4) (Compactness argument) Fix $\delta > 0$; ∂U cpt $\Rightarrow$ $\exists$ $\begin{cases} x_{0,i} \in \partial U, \ i = 1, \dots, N, \\ r_i > 0 \\ V_i := U \cap B(x_{0,i}, r_i/2) \\ v_i \in C^\infty(V_i) \end{cases}$

s.t. $\{B(x_{0,i}, r_i/2)\}_{i=1}^N$ covers ∂U and

$$\| v_i - u \|_{W^{k,p}(V_i)} \leq \delta .$$

Take open $V_0 \subset U$ s.t. $\{V_0\} \cup \{V_1, \dots, V_N\}$ is an open cover of U and choose $v_0 \in C^\infty(\bar{V}_0)$ s.t.

$$\| v_0 - u \|_{W^{k,p}(V_0)} \leq \delta .$$

5) (Partition of unity - again) Choose $\begin{cases} \zeta_i \in C_0^\infty(B(x_{0,i}, r_i/2)), \ i = 1, \dots, N \\ \zeta_0 \in C_0^\infty(V_0) \end{cases}$

s.t. $1 = \sum_{i=0}^N \zeta_i$.

Let $v = \sum_{i=0}^N \zeta_i v_i \in C^\infty(\bar{U})$ (Note: $\zeta_i v_i \in C^\infty(\bar{V}_i)$!) Hence $\forall \alpha, |\alpha| \leq k$,

$$\| D^\alpha v - D^\alpha u \|_{L^p(U)} \leq \sum_{i=0}^N \| D^\alpha (\zeta_i v_i - \zeta_i u) \|_{L^p(V_i)} \leq (N+1)\delta .$$

δ arb. $\Rightarrow$ claim. $\square$

<u>HW</u>: Rethink about the proof and check where did we use the fact that ∂U was C^1 ?
Could you improve on this proof ?

1.6. Extensions

Consider 1st the following trivial examples.

Let $U = \{x > 0\} \subset \mathbb{R}$. Assume $u \in W^{k,p}(U)$ and define

$$U_0 = \begin{cases} u, & x > 0 \\ 0, & x \leq 0 \end{cases} \quad (0\text{-extension})$$

$$U_{even}(x) = \begin{cases} u(x), & x > 0 \\ u(-x), & x \leq 0 \end{cases} \quad (\text{even extension})$$

$$U_{odd}(x) = \begin{cases} u, & x \geq 0 \\ -u(-x), & x < 0 \end{cases}$$

"the odd extension"

0-extension: ok for $k=0$ and $\|E_0 u\|_{L^p(\mathbb{R})} = \|u\|_{L^p(\mathbb{R}_+)}$.
Does not work for $k=1,2,\dots$, since jump discont $\Rightarrow$ no weak L^1_{loc}-derivative.

odd-extension: Also ok for $k=0$: $\|E_{odd} u\|_{L^p(\mathbb{R}^n)} = 2\|u\|_{L^p(\mathbb{R}_+)}$,
but not ok for $k=1,2,\dots$:
If $u(x) = e^{-x}$, then $u(-x) := -e^{x}$, and a origin this has a jump discontinuity $\leadsto$ no weak L^1_{loc}-derivative.

Even-extension: This is better: for simplicity consider $I=[0,1]$, $J=[-1,1]$.
We know that $C^\infty(\bar I)$ is dense in $W^{k,p}(I)$.
Let $u \in C^\infty(\bar I)$. Then Taylor $\Rightarrow$

$$u(x) = u_0 + u_1 x + O(x^2), \quad x \text{ near } 0.$$

$x<0$: $Eu(x) := E_{even} u(x) = u(-x) = u_0 - u_1 x + O(x^2)$

Then
$$\lim_{x\to 0+} u'(x) = u_1, \quad \lim_{x\to 0-} Eu'(x) = -u_1$$

So Eu is cont. across 0, dEu/dx has a jump discontinuity across 0
$\Rightarrow$ $Eu \in W^{1,p}(J)$ and $\|Eu\|_{W^{1,p}(J)} \le 2\|u\|_{W^{1,p}(J)}$.
This will not work in $W^{2,p}(J)$ anymore.

Let's try something more clever to also get the 2nd derivatives continuous:

For $x<0$ let

$$Eu(x) := \alpha_1 u(-\beta_1 x) + \alpha_2 u(-\beta_2 x), \quad \beta_i > 0$$

Cont. at 0: $u(0-) = \alpha_1 u(0+) + \alpha_2 u(0+) = u(0+) \Leftrightarrow \alpha_1 + \alpha_2 = 1$.
u' cont: $u'(0-) = -\alpha_1\beta_1 u'(0+) - \alpha_2\beta_2 u'(0+) = u'(0+) \Leftrightarrow -\alpha_1\beta_1 - \alpha_2\beta_2 = 1$
u'' cont: $u''(0-) = \alpha_1\beta_1^2 u''(0+) + \alpha_2\beta_2^2 u''(0+) = u''(0+) \Leftrightarrow \alpha_1\beta_1^2 + \alpha_2\beta_2^2 = 1$

Hence, to get $W^{2,p}$-extension (i.e. u' cont & a jump discont. for u'') we need

$$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_1\beta_1 + \alpha_2\beta_2 = -1 \end{cases}$$

Take now $\beta_1 = 1$, $\beta_2 = 2^{-1}$. Then

$$\begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_1 + \alpha_2/2 = -1 \end{cases} \iff \begin{cases} a_1 = -3 \\ a_2 = 4 \end{cases} \quad (\because)$$

One can play this game also for higher order derivatives:

$$Eu(x) = \sum_{j=0}^{N} \alpha_j u(-x/2^j) \quad \left(\text{or } \tilde{E}u(x) = \sum_{0}^{N} \tilde{\alpha}_j u(-2^j x)\right) \ldots \text{lots of possibilities.}$$

The extension of this to C^1 open sets of $\mathbb{R}^n$ is straightforward:

<u>Prop. 1.6.1</u> Assume $U \Subset \mathbb{R}^n$ is a bounded C^1-domain. Choose $V \Subset \mathbb{R}^n$ s.t.
$(1 \le p \le \infty)$ $\bar{U} \subset V$. Then $\exists$ bnd linear map

$$E : W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$$

s.t.

$$\begin{cases} Eu|_U = u \\ \operatorname{supp} Eu \subset V. \end{cases}$$

Pf. 1) Fix $x_0 \in \partial U$ and suppose $\exists$ open ball B with center x_0 s.t.

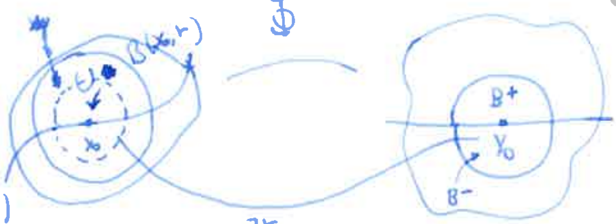
$$B^+ := B \cap \{x_n \ge 0\} \subset \bar{U}$$

$$B^-_n := B \cap \{x_n \le 0\} \subset \mathbb{R}^n \setminus U$$

2) Assume at 1st $u \in C^1(\bar{U})$. Define now

$$\bar{u}(x) := \begin{cases} u(x_1, \ldots, x_{n-1}, x_n), & x = (x_1, \ldots, x_n) \in B_+ \\ -3u(x_1, \ldots, x_{n-1}, -x_n) + 4u(x_1, \ldots, x_{n-1}, -x_n/2), & x = (x_1, \ldots, x_n) \in B_-. \end{cases}$$

3) We show that $\bar{u} \in C^1(B)$. Infact, in view of the 1-D remarks this is obvious (x_k, $k<n$ derivatives are cont. across $x_n = 0$ & normal derivatives match up).



$V = \Psi(B^+)$

$W = \Psi(B)$

depends only on constants chosen in def. of E

4) Also $\|\tilde{u}\|_{W^{1,p}(B)} \leq C \|u\|_{W^{1,p}(B^+)}$

γ C^1-func

5) "Straightening the boundary" Assume $U \cap B(x_0,r) = \{x \in B(x_0,r);\ x_n > \gamma(x_1,\dots,x_{n-1})\}$

Def. $\Phi : (x_1,\dots,x_n) \longmapsto (y_1,\dots,y_n)$, $\quad y_k = x_k$, $1 \leq k < n$

This is a C^1-diffeo near x_0 $\qquad y_n = x_n - \gamma(x_1,\dots,x_n)$

Here C^1 does not allow any better!

Hence $y_0 := \Phi(x_0)$ has a nbhd like in (1) & if $v(y) := u(\Psi(y))$, $\Psi = \Phi^{-1}$

Then extend v to $y_n < 0$ as $\tilde{v}$ like before.

$$\therefore \quad \|\tilde{v}\|_{W^{1,p}(B)} \leq C \|v\|_{W^{1,p}(B^+)}$$

$\Updownarrow$ change of variables & chain rule

$$\|\tilde{u}\|_{W^{1,p}(W)} \leq \hat{C} \|u\|_{W^{1,p}(U)}$$

6) Compactness of ∂U $\rightsquigarrow$ we can cover ∂U with open nbhds U_i, $i = 1,\dots,N$ s.t. (5) works on U_i, and we can extend $U_i \to W_i$.

Choose W_0 s.t.

$$V \subset W_0 \cup \bigcup_{i=1}^{N} W_i, \qquad \{\zeta_i\} \text{ a subordinate partition of unity.}$$

Let $\bar{u} = \sum_{i=0}^{N} \zeta_i \tilde{u}_i$, $\tilde{u}_0 = u$. Then by shrinking V if needed (or ζ_i's suitably) we have $\operatorname{supp} \bar{u} \subset V$ and

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(U)}, \quad u \in C^1(\bar{U}).$$

7) Assume now $u \in W^{1,p}(U)$. Choose $u_n \in C^\infty(\bar{U})$ s.t. $u_n \xrightarrow{W^{1,p}(U)} u$.

Then (u_n) is Cauchy in $W^{1,p}(U)$ $\Rightarrow$ $(\bar{u}_n)$ is Cauchy in $W^{1,p}(\mathbb{R}^n)$

$\Rightarrow$ $\exists$ lim $\bar{u} = \lim \bar{u}_n \in W^{1,p}(\mathbb{R}^n)$. This limit is ind. of the Cauchy seq. chosen; Def. $Eu := \bar{u}$. Then

$$\|Eu\|_{W^{1,p}(\mathbb{R}^n)} = \lim \|\bar{u}_n\|_{W^{1,p}} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}. \quad \square$$

Remark i) If γ is C^2, then the straight. of bndry argument is ok & we get also (remember that discont. in 2^{nd} normal deriv. is still ok!)

$$\|\bar{u}\|_{W^{2,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{2,p}(\mathbb{R}^n_+)}, \quad u \in C^\infty(\bar{U}).$$

Hence we also have an extension op $E: W^{2,p}(\mathbb{R}^n_+) \to W^{2,p}(\mathbb{R}^n)$.

ii) Higher order extension same way if ∂U is C^k and we use the trick outlined in the introduction!

1.7. Traces

Take $1 \leq p < \infty$.

Thm. 1.7.1 Assume $U \subseteq \mathbb{R}^n$ bnd, C^1-domain. Then $\exists$ bnd. linear operator

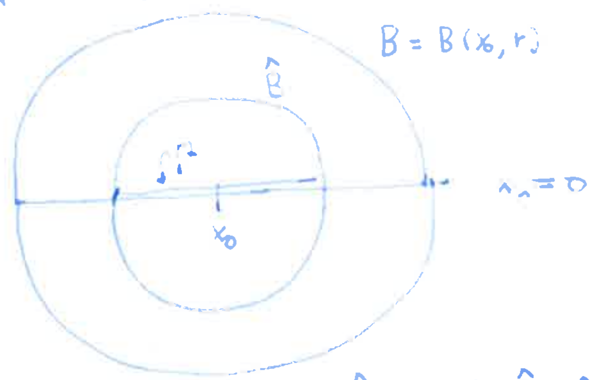
$$T: W^{1,p}(U) \longrightarrow L^p(\partial U)$$ w.r.t. surface measure dS

s.t.

(i) If $u \in W^{1,p}(U) \cap C(\bar{U})$, then $Tu = u|_{\partial U}$.

(ii) $\|Tu\|_{L^p(\partial\Omega)} \leq C_p' \|u\|_{W^{1,p}(U)}$

Pf. 1) (This is crucial argument) Let at first $u \in C^1(\bar{U})$. Let $x_0 \in \partial U$ and assume ∂U is again flat near x_0.



$\hat{B} = B(x_0, r/2)$

$B^+ = B \cap \{x_n \geq 0\}$

Choose $\zeta \in C_0^\infty(B)$, $\zeta \geq 0$ in B, $\zeta \equiv 1$ in $\hat{B}$; $\Gamma = \hat{B} \cap \partial U$.

Now we can estimate

$$\int_\Gamma |u|^p dx' \leq \int_{\{x_n = 0\}} \zeta |u|^p dx' \overset{\text{Green}}{=} -\int_{B^+} D_{x_n}(\zeta |u|^p)\, dx'$$

$$= -\int_{B^+} |u|^p (D_{x_n}\zeta) + \int p|u|^{p-1}(\operatorname{sgn} u) D_{x_n} u \, dx$$

$$\le C\Big[\int_{B^+} |u|^p dx + \int_{B^+} |u|^{p-1} |D_{x_n} u| \, dx$$

Young

$$\le C\Big[\int_{B^+} |u|^p dx + \frac{1}{q}|u|^{\overbrace{(p-1)q}^{=p}} + \frac{1}{p}|D_{x_n} u|^p \Big] dx$$

$$\le C \int_{B^+} |u|^p + |D_{x_n} u|^p \, dx .$$

Young's ineq:

$1 < p, q < \infty$

$\frac{1}{p} + \frac{1}{q} = 1$; then

$ab \le \frac{a^p}{p} + \frac{b^q}{q}$

Pf: e^x convex $\Rightarrow$

$ab = e^{\ln ab} = e^{\ln a + \ln b}$

$\le e^{\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q}$

$\le \frac{1}{p} e^{\ln a^p} + \frac{1}{q} e^{\ln b^q}$

$= \frac{1}{p} a^p + \frac{1}{q} b^q$

2) The general case follows by straightening the boundary (∂U is C^1 ~> we can differentiate once) & using a partition of unity. Then "extension by density" as before. □

The following Th. will justify the use of $W_0^{1,p}(U)$ as the space of vanishing Dirichlet data:

Th. 1.2.2 Assume U bdd & C^1-domain. Then

$$W_0^{1,p}(U) = \{ u \in W^{1,p}(U) ; \ Tu = 0 \}$$

Pf. Assume $u \in W_0^{1,p}(U)$. Then $\exists$ seq. (u_m), $u_m \in C_c^\infty(U)$ s.t. $u_m \xrightarrow{W^{1,p}} u \Rightarrow 0 = Tu_m \xrightarrow{L^p} Tu$.

The other direction takes more work; so assume

$$u \in W^{1,p}(U), \quad Tu = 0 .$$

Let's 1st prove the following:

$$\begin{cases} u \in W^{1,p}(\mathbb{R}^n_+), \ \operatorname{supp} u \subset \overline{\mathbb{R}^n_+} \text{ cpt} \\ Tu = 0 \text{ on } \mathbb{R}^{n-1} = \partial\mathbb{R}^n_+ . \end{cases}$$

$Tu = 0 \Rightarrow \exists$ sequence $u_m \in C^1(\overline{\mathbb{R}^n_+})$ (with cpt support) s.t.

$$\begin{cases} u_m \to u & \text{in } W^{1,p}(\mathbb{R}^n_+) \\ Tu_m = u_m|_{\mathbb{R}^{n-1}} \to 0 & \text{in } L^p(\mathbb{R}^{n-1}). \end{cases}$$

Let $x' \in \mathbb{R}^{n-1}$, $x_n \ge 0$. Then

$$|u_m(x', x_n)| \le |u_m(x', 0)| + \int_0^{x_n} |D_{x_n} u_m(x', t)| \, dt$$

$\Rightarrow$

$$\int_{\mathbb{R}^{n-1}} |u_m(x', x_n)|^p dx' \le C\Big(\int_{\mathbb{R}^{n-1}} |u_m(x', 0)|^p dx'$$

$$+ \int_{\mathbb{R}^{n-1}} \Big| \int_0^{x_n} u_m(x', t) \, dt \Big|^p dx' \Big) \qquad s = \frac{t}{x_n}, \ 0 \le s \le 1, \quad dt = x_n ds$$

$$= C\Big(\int_{\mathbb{R}^{n-1}} |u_m(x', 0)|^p dx' + \int_{\mathbb{R}^{n-1}} \Big| x_n \int_0^1 u_m(x', s) \, ds \Big|^p dx' \Big)$$

Jensen

$$\le C\Big(\int_{\mathbb{R}^{n-1}} |u_m(x', 0)|^p dx' + x_n^p \int_{\mathbb{R}^{n-1}} \int_0^1 |u_m(x', s)|^p ds \, dx' \Big)$$

$s = t x_n$

$$\underset{+\text{Fubini}}{=} C\Big(\int_{\mathbb{R}^{n-1}} |u_m(x', 0)|^p dx' + x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |u_m(x', t)|^p dt \, dx' \Big)$$

Letting $m \to \infty$ we get

$$\Big[\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' \le C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} |u(x', t)|^p dt \, dx' \Big]$$

for a.e. x_n

(*)

Choose now $\zeta \in C^\infty(\mathbb{R}_+)$ s.t.

$\zeta = 1$ on $[0,1]$, $\zeta = 0$ on $\mathbb{R}_+ \setminus [0,2]$,

$$0 \le \zeta \le 1$$

Let

$\zeta_m(x) = \zeta(m x_n)$, $x \in \mathbb{R}^n_+$

$w_m = u(1-\zeta_m)$ "supported at $\{x_n \ge \frac{1}{m}\}$"

Then

$$D_{x_n} w_m = (D_{x_n} u)(1-\zeta_m) - \zeta_m' m u_m$$

$$D_{x'} w_m = (D_{x'} u)(1-\zeta_m)$$

Hence

$$\int_{\mathbb{R}^n_+} |Dw_m - Du|^p dx \le \int_{\mathbb{R}^n_+} |\overbrace{D_{x'}(w_m - u)}^{-\zeta_m D_{x'} u}|^p dx$$

$$+ \int_{\mathbb{R}^n_+} |\underbrace{D_{x_n} w_m - D_{x_n} u}_{= -\zeta_m D_{x_n} u - \zeta_m' m u m}|^p dx$$

integrate ζ_m'
ζ from integrals

$$\le C \underbrace{\int_{\mathbb{R}^n_+} |\zeta_m|^p |D_x u|^p dx}_{=: A} + \underbrace{C m^p \int_0^{2/m} \int_{\mathbb{R}^n_+} |u|^p dx' dt}_{=: B}.$$

Now $\operatorname{supp} \zeta_m \subset [0, 2/m]$, so

$$A \to 0 \quad \text{as } m \to \infty.$$

On the other hand:

$$B \le C m^p \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u|^p dx' dt$$

$$\overset{(*)}{\le} C m^p \int_0^{2/m} t^{p-1} \int_0^{\le t} \int_{\mathbb{R}^{n-1}} |u(x', s)|^p ds\, dx'$$

$$\le C m^p \underbrace{\left(\int_0^{2/m} t^{p-1} dt\right)}_{\frac{1}{p}\left(\frac{2}{m}\right)^p} \left(\int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' dx_n\right)$$

$$\le C \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' dx_n \xrightarrow[m \to \infty]{} 0.$$

$\therefore$ Hence $w_m \to u$ in $W^{1,p}$, $\operatorname{supp} w_m \subset \{x_n \ge \frac{1}{m}\}$

So we can mollify w_m s.t. $w_{m,\varepsilon} \in C_0^\infty(\mathbb{R}^n_+)$

and $w_{m,\varepsilon} \to u$ in $W^{1,p}(\mathbb{R}^n_+)$.

$\therefore u \in W_0^{1,p}(\mathbb{R}^n_+)$.

The general case - as usual by mov - using "straightening of bnd" & partitions of unity. ∎

1.8. ~~Sobolev's eq.~~ Gagliardo-Nirenberg-Sobolev ineq.

Palautetaan mieliin kaksi asiaa:

• Esim, jossa osoitettiin että $p < n \Rightarrow$ in $W^{1,p}(U)$ ~~...~~, UPS - badly Esimerkki - there are functions which are unbounded in a dense countable subset.

• Ex 4/HW2: $u \in W^{1,p}(\mathbb{R}) \Rightarrow$

$$|u(x) - u(y)| \le |x-y|^{1-\frac{1}{p}} \int_x^y |u'|^p dt$$

Hence $p > 1 \Rightarrow u$ is Hölder with exponent $\gamma = 1 - 1/p$
$p = 1 \Rightarrow u$ is continuous with modulus of continuity $\sim \int_x^y |u'|^p \, dt$

If we go back to case $p < n$ one could ask if the singularities of $W^{1,p}$ function could still be L^q integrable for some q. This leads to so called Gagliardo-Nirenberg-Sobolev ineq:

Let's ask if, given p, $\exists\, q$ s.t.

$$(\because) \qquad \|u\|_{L^q(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)} \qquad \forall u \in C_0^\infty(\mathbb{R}^n)$$

of course - independent of u.

There is a very clever and easy argument to find the correct q: for $\lambda > 0$, let

$$u_\lambda(x) = u(\lambda x).$$

Then if $(\because)$ holds, it must hold for u_λ $\forall \lambda$ with C independent of λ. Let's check

$$\|u_\lambda\|^q_{L^q(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |u_\lambda|^q dx = \int_{\mathbb{R}^n} |u(\lambda x)|^q dx$$

$y = \lambda x$, $dx = \lambda^{-n} dy$

$$= \lambda^{-n} \int_{\mathbb{R}^n} |u|^q dx = \|u\|^q_{L^q(\mathbb{R}^n)} / \lambda^n .$$

On the other hand,

$$Du_\lambda(x) = \lambda Du(\lambda x),$$

and hence

$$\int_{\mathbb{R}^n} |Du_\lambda(x)|^p dx = \lambda^p \int_{\mathbb{R}^n} |(Du)(\lambda x)|^p dx = \lambda^{p-n} \|Du\|^p_{L^p(\mathbb{R}^n)}$$

$\therefore$ $\forall \lambda > 0$ we must have - if $(\because)$ holds -

$$\left(\lambda^{-n} \|u\|^q_{L^q(\mathbb{R}^n)}\right)^{1/q} \leq C\left(\lambda^{p-n} \|Du\|^p_{L^p(\mathbb{R}^n)}\right)^{1/p} \quad \forall \lambda > 0$$

$$\Leftrightarrow \quad \|u\|_{L^q(\mathbb{R}^n)} \leq C \lambda^{\frac{p-n}{p} + \frac{n}{q}} \|Du\|_{L^p(\mathbb{R}^n)} \quad \forall \lambda > 0.$$

Hence

$$0 = 1 - \frac{n}{p} + \frac{n}{q} \iff \frac{n}{q} = \frac{n}{p} - 1 = \frac{n-p}{p}$$

$$\iff q = \frac{np}{n-p}$$

Let $\begin{cases} p^* := \dfrac{np}{n-p} & \text{"the Sobolev-conjugate of } p\text{"} \\ \text{i.e.} & \\ \dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{1}{n} & p^* > p, \quad 1 \leq p < n \end{cases}$

Now we have

<u>Th. 1.8.1</u> (Gagliardo-Nirenberg-Sobolev ineq.)
Let $1 \leq p < n$. Then $\exists\, C = C_{p,n}$ s.t.

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)} \qquad \forall u \in C_0^1(\mathbb{R}^n)$$

Trivial obs: $u \equiv \text{const} \neq 0$ shows that something needs to be assumed of the support.

Pf. 1) Assume $p=1$. Since supp u cpt,

$$u(x) = \int_{-\infty}^{x_i} (D_{x_i} u)(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)\, dy_i$$

$$\Rightarrow |u(x)| \leq \int_{-\infty}^{\infty} |D_{x_i} u(x_1, \dots, y_i, \dots, x_n)|\, dy_i$$

$$\Rightarrow |u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}} |(D_{x_i} u)(x_1, \dots, y_i, \dots, x_n)|\, dy_i \right)^{\frac{1}{n-1}}$$

Note: $1^* = \frac{n}{n-1}$

Hence

$$\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}} dx_1 \leq \int_{\mathbb{R}} \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(y)|\, dy_i \right)^{\frac{1}{n-1}} dx_1$$

$$= \left(\int_{\mathbb{R}} |Du|\, dy_1 \right)^{1/n-1} \left(\int_{\mathbb{R}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} |Du|\, dy_i \right)^{\frac{1}{n-1}} dx_1 \right)$$

$$\overset{*}{\leq} \left(\int_{\mathbb{R}} |Du|\, dy_1 \right)^{\frac{1}{n-1}} \left(\prod_{2}^{n} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |Du|\, dy_i\, dx_1 \right)^{\frac{1}{n-1}}.$$

By applying General Hölder with $u_i = \left(\int_{\mathbb{R}} |Du|\, dy_i \right)^{\frac{1}{n-1}}, i = 2, \dots, n$, $p_i = n-1$

*) General Hölder: $u_i \in L^{p_i}(\mathbb{R}^n)$, $i = 1, \dots, m$, $\frac{1}{p_1} + \dots + \frac{1}{p_m} = 1$.

Then $\int |u_1 \cdots u_m|\, dx \leq \prod^{m} \|u_i\|_{L^{p_i}(\mathbb{R}^n)}$

Pf. HW.

Integrate this w.r.t. x_2 (shuffle the order)

$$\iint_{\mathbb{R}^2} |u|^{\frac{n}{n-1}} dx_1 dx_2 \leq \left(\int_{\mathbb{R}^2} |Du|\, dx_1 dy_2 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \prod_{i=1, i \neq 2}^{n} I_i^{\frac{1}{n-1}} dx_2,$$

where

$I_1 := \int |Du|\, dy_1$, $I_i = \iint |Du|\, dx_1 dy_i$.

Apply general Hölder again:

$$\iint |u|^{\frac{n}{n-1}} dx_1 dx_2 \leq \left(\int_{\mathbb{R}^2} |Du|\, dx_1 dy_2 \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}^2} |Du|\, dy_1 dx_2 \right)^{\frac{1}{n-1}} \times \prod_{i=3}^{n} \left(\int |Du|\, dx_1 dx_2 dy_i \right)^{\frac{1}{n-1}}$$

Repeat ...

$$\iint \cdots \int |u|^{\frac{n}{n-1}} dx \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |Du|\, dx_1 \cdots dy_i \cdots dx_n \right)^{\frac{1}{n-1}}$$

$$= \left(\int |Du|\, dx \right)^{\frac{n}{n-1}}$$

$\therefore \quad \|u\|_{L^{1^*}(\mathbb{R}^n)} \leq \|Du\|_{L^1(\mathbb{R}^n)}$.

2) Let now $1 < p < n$. Let now

$v := |u|^{\gamma}$, where we fix γ later.

Then apply 1) to v:

$$\left(\int |u|^{\frac{\gamma n}{n-1}} dx \right)^{\left(\frac{n}{n-1}\right)^{-1}} = \left(\int |v|^{\frac{n}{n-1}} dx \right)^{\left(\frac{n}{n-1}\right)^{-1}} \leq \int |Dv|\, dx = \int |D|u|^{\gamma}|\, dx.$$

$$= \gamma \int |u|^{\gamma-1} \underbrace{D|u|}_{=(\mathrm{sgn}\, u) Du} dx \leq \gamma \int |u|^{\gamma-1} |Du|\, dx$$

Hölder $\frac{p-1}{p}$

$$\leq \gamma \left(\int |u|^{(\gamma-1)p/p-1} dx \right)^{\frac{p-1}{p}} \left(\int |Du|^p dx \right)^{1/p}.$$

Choose γ s.t.

$$\frac{(\gamma-1)p}{p-1} = \frac{\gamma n}{n-1} \iff \gamma \Big[\underbrace{\frac{n}{n-1} - \frac{p}{p-1}}_{=\frac{(p-1)n - p(n-1)}{(n-1)(p-1)}} \Big] = \frac{-p}{p-1}$$

$$\iff \gamma = \frac{-p}{\cancel{p-1}} \cdot \frac{(n-1)\cancel{(p-1)}}{p-n}$$

$$= \frac{p(n-1)}{n-p} > 1 \text{ since } 1 < p < n.$$

Then

$$\frac{\gamma n}{n-1} = \frac{p(n-1)}{n-p} \cdot \frac{n}{n-1} = \frac{np}{n-p} = p^*$$

Also:

$$\frac{(\gamma-1)p}{p-1} = p^* \iff \frac{\gamma-1}{p^*} = \frac{p-1}{p}$$

and we get

$$\left(\int |u|^{p^*} dx \right)^{\gamma/p^*} \leq \gamma \left(\int |u|^{p^*} \right)^{(\gamma-1)/p^*} \left(\int |Du|^p dx \right)^{1/p}$$

$$\iff \|u\|_{L^{p^*}(\mathbb{R}^n)} \leq \gamma \left(\int |Du|^p dx \right)^{1/p} = \gamma \|Du\|_{L^p(\mathbb{R}^n)}. \quad \square$$

We can also get a similar estimate on bnd. domains for $W^{1,p}$-norm:

Th. 1.8.2. $U \subset \mathbb{R}^n$ bnd C^1-domain. Assume $1 \leq p < n$.

Then

$$\|u\|_{L^{p^*}(U)} \leq C \|u\|_{W^{1,p}(U)}$$

C depends only on n, p & U.

Pf. Let $E: W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ be the extension op s.t. $\operatorname{supp} Eu \subset K$ $\forall u$, $K \subset\subset \mathbb{R}^n$ fixed.

Let $\bar{u} = Eu$; $\operatorname{Supp} \bar{u}$ cpt $\Rightarrow$ $\exists$ sequence $u_m \in C_0^\infty(\mathbb{R}^n)$ s.t.

$$u_m \longrightarrow \bar{u} \text{ in } W^{1,p}(\mathbb{R}^n).$$

Now C-~~N~~-S $\Rightarrow$

$$\|u_m - u_\ell\|_{L^{p^*}(\mathbb{R}^n)} \leq \|D(u_m - u_\ell)\|_{L^p(\mathbb{R}^n)}$$

$$\leq \|u_m - u_\ell\|_{W^{1,p}(\mathbb{R}^n)}$$

$\therefore$ (u_m) Cauchy in $L^{p^*}(\mathbb{R}^n)$

and hence also $u_m \to \bar{u}$ in $L^{p^*}(\mathbb{R}^n)$, $\bar{u} \in L^{p^*}(\mathbb{R}^n)$.

Since

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du_m\|_{L^p(\mathbb{R}^n)} \quad \forall m$$

$$\Rightarrow \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|D\bar{u}\|_{L^p(\mathbb{R}^n)}$$

$\therefore$

$$\|u\|_{L^{p^*}(U)} \leq \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|D\bar{u}\|_{L^p(\mathbb{R}^n)}$$

$$\leq C \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C' \|u\|_{W^{1,p}(U)}. \quad \square$$

If we also know $u \in W_0^{1,p}(U)$, then we can ignore the $\|u\|_{L^p(U)}$ from $W^{1,p}(U)$-norm:

Th. 1.8.3. $U \Subset \mathbb{R}^n$ bounded, $1 \le p < n$. Assume $u \in W_0^{1,p}(U)$. Then $\forall\ q \in [1, p^*]$ we have

$$\|u\|_{L^q(U)} \le C^U_{p,n,q} \|Du\|_{L^p(U)}$$

Since $p \le p^*$, we especially have

$$\|u\|_{L^p(U)} \le C^U_{p,n} \|Du\|_{L^p(U)}.$$

Pf. $u \in W_0^{1,p}(U) \Rightarrow \exists$ sequence $u_m \in C_0^\infty(U)$ s.t. $u_m \xrightarrow{W^{1,p}} u$. Extend u_m to $\mathbb{R}^n$ as zero. Then

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le C \|Du_m\|_{L^p(\mathbb{R}^n)}$$

Also

$$\|u_m - u_n\|_{L^{p^*}(U)} = \|u_m - u_n\|_{L^{p^*}(\mathbb{R}^n)} \le C \|D(u_m - u_n)\|_{L^p(\mathbb{R}^n)} \le C \|u_m - u_n\|_{W^{1,p}(U)}$$

$\Rightarrow (u_m)$ is Cauchy in $L^{p^*}(U)$. Hence we have $u \in L^{p^*}(U)$ and

$$\|u\|_{L^{p^*}(U)} \le C \|Du\|_{L^p(U)}.$$

Since U bnd, $p \le p^* \Rightarrow \|u\|_{L^p(U)} \le C_{p,p^*} \|u\|_{L^{p^*}(U)}$ (Thm. ?) the other claim follows. $\square$

This is often called the (easy version) of Poincaré's inequality.

Note: U bnd, then in $W_0^{1,p}(U)$ $\|Du\|_{W^{1,p}(U)}$ defines a norm that is equivalent to $W^{1,p}$-norm.

1.9. Morrey's inequality

Suppose now $n < p \le \infty$.

Th. 1.9.1 (Morrey's ineq.) $\exists$ const. $C = C_{p,n}$ s.t.

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C \|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall u \in C^1(\mathbb{R}^n),$$

where $\gamma = 1 - n/p$.

Pf. 1) Choose a ball $B(x,r) \subset \mathbb{R}^n$.

We 1st prove that $\exists\ C_n' = C$ = constant s.t.

(∗) $$\fint_{B(x,r)} |u(y) - u(x)|\, dy \le C \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, dy$$

Now, let $|w| = 1$. Then $\forall\ 0 < s < r$:

$$|u(x+sw) - u(x)| = \left| \int_0^s \frac{d}{dt} u(x+tw)\, dt \right|$$

$$= \left| \int_0^s \langle Du(x+tw), w \rangle\, dt \right|$$

$$\le \int_0^s |Du(x+tw)|\, dt$$

So,

$$\int |u(x+sw) - u(x)|\, dS(w) \le \int_0^s \int_{|w|=1} |Du(x+tw)|\, dt\, dS$$

$$= \int_0^s \int_{|w|=1} \frac{|Du(x+tw)|}{t^{n-1}} \underbrace{t^{n-1}\, dt\, dS}_{= dy}, \quad y = x + tw$$

$$\overset{\text{Polar}}{\underset{\text{coord}}{=}} \int_{B(x,s)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy \quad \text{, since } t = |tw| = |x-y|$$

$$\leq \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy .$$

Hence

$$\int_{|w|=1}\int_0^r |u(x+sw)-u(x)|\, s^{n-1}\,dS(w) \leq \underbrace{\frac{r^n}{n}}_{\int_0^r s^{n-1}ds} \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy$$

Let $y = x+sw$. Then above ineq. gives (ii) | Note $|B(x,r)| \simeq r^n$

1) Fix $x \in \mathbb{R}^n$. Then

$$|u(x)| = \fint_{B(x,1)} |u(x)|\,dy \leq \fint_{B(x,1)} |u(y)-u(x)|\,dy + \fint_{B(x,1)} |u(y)|\,dy$$

$$\overset{(ii)}{\leq} C\int_{B(x,1)} \frac{|Du(y)|}{|x-y|^{n-1}}\,dy + \underbrace{C}_{\text{dep. on } p}\|u\|_{L^p(B(x,1))}$$

$$\overset{\text{(Hölder)}}{\leq} C\left(\int_{B(x,1)} |Du(y)|^p\,dy\right)^{1/p}\left(\int_{B(x,1)} \frac{dy}{|x-y|^{(n-1)p/(p-1)}}\right)^{\frac{p-1}{p}} + C\|u\|_{L^p(\mathbb{R}^n)}$$

Now

$$\frac{(n-1)p}{p-1} - n = \frac{n-p}{p-1} < 0 \quad \text{since } n < p$$

so

$$\int_{B(x,1)} \frac{dy}{|x-y|^{(n-1)p/(p-1)}} < \infty$$

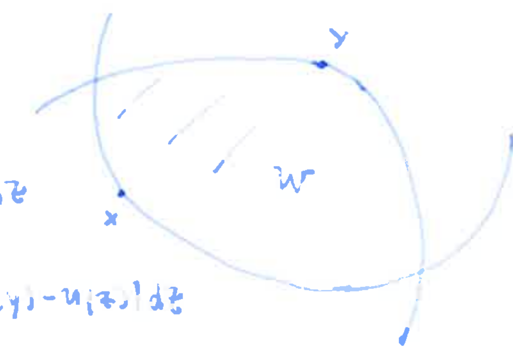
and we have shown

$$|u(x)| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

$$\Rightarrow \sup_x |u| \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} . \qquad (iii)$$

3) Finally, let $x, y \in \mathbb{R}^n$, let $r = |x-y|$ and

$W = B(x,r) \cap B(y,r)$



Then

$$|u(x)-u(y)| = \fint_W |u(x)-u(y)|\,dz$$

$$\leq \fint_W |u(x)-u(z)|\,dz + \fint_W |u(y)-u(z)|\,dz$$

Let's estimate these terms:

$$\fint_W |u(x)-u(z)|\,dz \leq C\fint_{B(x,r)} |u(x)-u(z)|\,dz$$

$$\underset{\text{+ Hölder}}{\overset{(ii)}{\leq}} C\left(\int_{B(x,r)} |Du(z)|^p\,dz\right)^{1/p}\left(\int_{B(x,r)} \frac{dz}{|x-z|^{(n-1)p/(p-1)}}\right)^{\frac{p-1}{p}}$$

$$\leq C\left(r^{\overbrace{n-(n-1)p/(p-1)}^{=\frac{p-n}{p-1}}}\right)^{1-\frac{1}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

$$= C r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)} .$$

Similarly for $\fint_W |u(y)-u(z)|\,dz$.

Hence

$$|u(x)-u(y)| \le C r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)} = C|x-y|^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

$\therefore$ Above & (ii) $\Rightarrow$

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|Du\|_{L^p(\mathbb{R}^n)}. \quad \square$$

Def. u^* is a version of u if $u = u^*$ a.e.

$\therefore$

Th. 1.9.2 Assume $U \subset \mathbb{R}^n$ bnd C^1-domain, $n < p \le \infty$. If $u \in W^{1,p}(U)$, then u has a version $u^* \in C^{0,\gamma}(U)$, $\gamma = 1-\frac{n}{p}$ s.t.

$$\|u^*\|_{C^{0,\gamma}(U)} \le C\|u\|_{W^{1,p}(U)}.$$

From now on we identify u^* and u

Pf. Let $E: W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$ be the extension op, $\bar{u} = Eu$ s.t. supp $Ev \subset K$, $U \subset K$ & K bnd $\forall\, v \in W^{1,p}(U)$.

Assume $n < p < \infty$; let $u_m \xrightarrow{W^{1,p}(\mathbb{R}^n)} \bar{u}$, $u_m \in C_0^\infty(K)$.

Then

$$\|u_m - u_\ell\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|u_m - u_\ell\|_{W^{1,p}(\mathbb{R}^n)}$$

Cauchy

$\Rightarrow \exists\, u^* \in C^{0,\gamma}(\mathbb{R}^n)$ s.t.

$$u_m \longrightarrow u^* \text{ in } C^{0,\gamma}(\mathbb{R}^n),$$

and

$$\|u^*\|_{C^{0,\gamma}(U)} \le \|u^*\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \le \tilde{C}\|u\|_{W^{1,p}(U)}.$$

$\square$

These can be generalized as in [Evans, Th. 6 in 5.6], but we will prove them if needed. Not now.

1.10 Compactness

Let's briefly recall the def. & basic properties of compact ops in Banach spaces.

Remember 1st that in $\mathbb{R}^n$ a set $F \subset \mathbb{R}^n$ is compact iff it is closed and bounded. This is not true anymore in infinite dimensional case.

Ex. $H = \ell^2$, $e_n = (0, \dots, \overset{(n)}{1}, \dots 0, \dots)$, $n = 1, 2, \dots$, the standard ON basis. Then

$$\|e_n\| = 1, \quad \langle e_n, e_m \rangle = 0 \text{ if } n \ne m \text{ and}$$

hence

$$\|e_n - e_m\| = \langle e_n - e_m, e_n - e_m \rangle^{\frac{1}{2}} = \big(2 - 2\underbrace{\langle e_n, e_m \rangle}_{=0}\big)^{\frac{1}{2}} = \sqrt{2}.$$

Hence $(e_n)_{n=1}^\infty$ does not have a convergent subsequence $\Rightarrow \{x \in \ell^2;\ \|x\| = 1\}$ is a bnd closed set which is not compact.

Especially, if X, Y Banach and $A: X \to Y$ bnd linear, image of a closed, bnd subset may not be precompact i.e. $\overline{A(F)}$ not (generally) cpt, if F closed & bnded.

Def. A bnd linear map $A : X \to Y$ (Banach) is <u>compact</u> if $\overline{A(F)}$ is compact in Y $\forall$ F closed, bounded.

<u>Note</u> Enough to take $F = \overline{B(0,1)}$ = the closed unit ball of E.

<u>Easy corollaries</u>: i) $A_1, A_2 : X \to Y$ cpt $\Rightarrow$ $\begin{cases} \alpha A_1 \text{ is cpt}, \alpha \in \mathbb{C} \\ A_1 + A_2 \text{ is cpt.} \end{cases}$

[Pf. Consider subsequences]

ii) $A : X \to Y$, $B : Y \to Z$ bnd linear.
If either A or B is cpt, then BA is also.

iii) Hence compact ~~lin~~ maps $K : X \to X$ form an ideal $= \mathcal{K}(X)$.

iv) If K_n are cpt, $K_n : X \to Y$ and $\|K - K_n\| \to 0$, (op. norm) then K is also cpt (Pf. Cantor-sequence of subsequences)

v) Hence $\mathcal{K}(X)$ is a (norm) closed ideal of $\mathcal{L}(X)$.

vi) All finite dimensional linear maps $F : X \to Y$ are cpt.
$\Updownarrow$ def
$F(X) \subset Y$ is a finite dimensional subspace.

vii) All norm limits of ~~cpt ops~~ finite dimensional ops are cpt.

<u>Injoed</u>: if X, Y are Hilbert: K cpt $\Leftrightarrow$ $\exists$ fin.dim ops F_n s.t. $\|F_n - K\| \to 0$. <u>Not</u> true in general Banach spaces.

<u>Ex.</u> $X = Y = L^2[0,1]$.

$Kf(x) = \int_0^1 k(x,t) f(t)\,dt$, $k \in C[0,1]^2$.

Then K cpt.

<u>Pf.</u> Let $I^{(n)}_{r,s} = \underbrace{\left[\frac{r}{n}, \frac{r+1}{n}\right)}_{J^{(n)}_r} \times \underbrace{\left[\frac{s}{n}, \frac{s+1}{n}\right)}_{= J^{(n)}_s}$, $0 \le r < n$, $0 \le s < n$ integers

Choose $(x^{(n)}_{r,s}, t^{(n)}_{r,s}) \in I^{(n)}_{r,s}$.

~~$K^{(n)} f = \ldots k(x^{(n)}_{r,s}$~~

$Kf(x) = \sum_{r,s} \int \chi_{J^{(n)}_r}(x) \chi_{J^{(n)}_s}(t) \, k(x,t) f(t)\,dt$

$\underset{\text{sum}}{\overset{\text{Riemann}}{\sim}} \underbrace{\sum_{r,s} \int \chi^{(n)}_r(x) \chi^{(n)}_s(t)\, k(x^{(n)}_{r,s}, t^{(n)}_{r,s}) f(t)\,dt}$

$= : K^{(n)} f$.

Then $K^{(n)}$ fin.dim, $\|K - K^{(n)}\| \to 0$ (def. of Riemann int.)

$\Rightarrow$ claim. $\square$

This is a classical ex. of a cpt linear (integral) op.

~~Def~~ Next we consider when the imbedding map

$W^{1,p}(U) \to L^q(U)$ (if indeed defined)

~~to an imbedding~~ is cpt.

We will prove compactness of embeddings by extracting a convergent subsequence from bounded sequence. The classical tool facilitating this is the Ascoli-Arzela Thm:

<u>Ascoli-Arzela</u>: Let $V \subset \mathbb{R}^d$ be cpt, and

$$f_n : \mathbb{R}^n \to \mathbb{C}, \quad n = 1, \dots,$$

be a sequence of functions that is

a) uniformly bounded on K i.e. $\exists$ constant $M > 0$ s.t.

$$|f_n(x)| \leq M \quad \forall x \in K, \ m \in \{1, 2, \dots\}$$

b) equicontinuous on K i.e. $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$|f_\ell(x) - f_\ell(y)| < \varepsilon \quad \text{when } |x-y| < \delta \quad \forall \ell \in \{1, \dots\}.$$

Then $\exists$ subsequence (f_{n_k}) converging uniformly on K.

For proof see any textbook on real analysis or functional analysis.

<u>Th. 1.10.1</u> (Rellich-Kondrachov / compactness embedding thm.)

Assume $U \Subset \mathbb{R}^n$ is a bounded C^1-domain, $1 \leq p < n$. Then

$$W^{1,p}(U) \subset L^q(U)$$

is compact $\forall \ 1 \leq q < p^*$.

Pf. <u>Step I</u>: By the prev. results, the embedding $W^{1,p}(U) \to L^q(U)$ is continuous, $1 \leq q < p^*$.

Hence it remains to prove compactness; assume $(u_m)_{m=1}^\infty$ be a bounded seq. in $W^{1,p}$. Enough to find a subs. conv. ~~weakly~~ in $L^q(U)$.

<u>Step II</u> Let $E : U \to \mathbb{R}^n$ be the extension op.

Assume V a fixed bound. nbhd of U and $\operatorname{supp} Eu \subset V \ \forall u \in W^{1,p}$. Then $\tilde{u}_m := Eu_m$ is bnd in $W^{1,p}(\mathbb{R}^n)$, $\operatorname{supp} \tilde{u}_m \subset V$. Enough to find a subseq. $(\tilde{u}_{m_k})$ conv. in $L^q(V)$, $1 \leq q < p^*$.

<u>Step III</u> Let $u^\varepsilon_m := \eta_\varepsilon * \tilde{u}_m$ and choose $\varepsilon > 0$ so small that $\operatorname{supp} u^\varepsilon_m \subset V$.

<u>Step IV</u> Let's 1st study the convergence of u^ε_m to $\tilde{u}_m$ in $L^q(V)$ as $\varepsilon \to 0$. Assume first that also $\tilde{u}_m$ is smooth: then

$$u^\varepsilon_m(x) - \tilde{u}_m(x) = \int_{B(0,1)} \eta(y)(u_m(x - \varepsilon y) - u_m(x))\, dy$$

$$= \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt} u_m(x - \varepsilon t y)\, dt\, dy$$

$$= -\varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \langle Du_m(x - \varepsilon t y), y \rangle\, dt\, dy$$

$\therefore$ Fubini $\Rightarrow$

$$\int_V |u^\varepsilon_m(x) - \tilde{u}_m(x)|\, dx \leq \varepsilon \int_{B(0,1)} \eta(y) \int_V \int_0^1 |D\tilde{u}_m(x - \varepsilon t y)|\, dt\, dy\, dx$$

$$\leq \varepsilon \int_V |D\tilde{u}_m|\,dx$$

And by density of $C^\infty(\bar V)$ this holds also if $\hat u_m \in W^{1,p}(V)$. Hence

$$\|u^\varepsilon_m - \tilde u_m\|_{L^1(V)} \leq \varepsilon \|D\hat u_m\|_{L^1(V)} \leq \varepsilon \overbrace{\|D\hat u_m\|_{L^p(V)}}^{\leq C}$$

$\uparrow$ V bnd.

$\therefore$ $u^\varepsilon_m \longrightarrow \tilde u_m$ in $L^1(V)$ uniformly in m!

Now recall the Riesz-Thorin interpolation/convexity thm:

(RT) $1 \leq s \leq r \leq t \leq \infty$, $\frac{1}{r} = \frac{\theta}{s} + \frac{(1-\theta)}{t}$. Then $\forall\, u \in L^s(U) \cap L^t(U)$:

$$\|u\|_{L^r(U)} \leq \|u\|^{\theta}_{L^s(U)} \|u\|^{1-\theta}_{L^t(U)}$$

Since $1 \leq q < p^*$ we have

$$\frac{1}{q} = \theta + \frac{(1-\theta)}{p^*}, \quad \theta \neq 0,$$

and thus

$$\|u^\varepsilon_m - \tilde u_m\|_{L^q(V)} \leq \|u^\varepsilon_m - \tilde u_m\|^{\theta}_{L^1(V)} \underbrace{\|u^\varepsilon_m - \hat u_m\|^{1-\theta}_{L^{p^*}(V)}}_{\text{G-N-S}}$$

$$\leq C \|u^\varepsilon_m - \hat u_m\|_{W^{1,p}(V)}$$ (bnd.)

$$\leq C \|u^\varepsilon_m - \hat u_m\|^{\theta}_{L^1(V)}$$

$$\leq C\varepsilon^{\theta}$$

$\therefore$ $u^\varepsilon_m \longrightarrow \tilde u_m$ in $L^q(V)$ uniformly in m.

Step V: We need prove that the sequence (u^ε_m) is unif. bounded and equicontinuous.

Now

$$|u^\varepsilon_m(x)| \leq \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\,|u_m(y)|\,dy$$

$$\leq \|\eta_\varepsilon\|_{L^\infty} \|\tilde u_m\|_{L^1(V)} \leq C\varepsilon^{-n}$$, C ind. of m since u_m was bnd in $W^{1,p}$!

Similarly,

$$|Du^\varepsilon_m(x)| \leq \|D\eta_\varepsilon\|_{L^\infty} \|\hat u_m\|_{L^1(V)} \leq C\varepsilon^{-(n+1)}$$

Thus M.V.T $\Rightarrow$ u^ε_m is unif. bnd and equicont.

Step VI: Fix $\delta > 0$. We show that $\exists$ subseq $\{u_{m_j}\}$ s.t.

$$\limsup_{j,k} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta.$$

Choose 1st $\varepsilon > 0$ so small that

$$\|u^\varepsilon_m - \tilde u\|_m\| < \delta/2, \quad m = 1, 2, \ldots$$

Ascoli-Arzela $\Rightarrow$ $\exists$ subsequence $(u^\varepsilon_{m_j})$ s.t. it converges uniformly on $\bar V$

$$\therefore \limsup_{j,k} \|u^\varepsilon_{m_j} - u^\varepsilon_{m_k}\| = 0$$

$$\begin{cases} H^s(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) ;\ \hat{f} \in L^1_{loc} \text{ and } \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < \infty \} \\ \|f\|^2_{H^s(\mathbb{R}^n)} = \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \end{cases}$$

This is a Hilbert-space with ~~norm~~ inner-product

$$(f,g)_{H^s(\mathbb{R}^n)} = \int (1+|\xi|^2)^s \hat{f}(\xi) \overline{\hat{g}(\xi)}\, d\xi .$$

Also, if we know ~~supp f ⊂~~ $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ is periodic with period 2π we can define the α^{th} Fourier cf

$$\hat{f}(\alpha) = \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} e^{-i\langle x,\alpha\rangle} f(x)\, dx, \quad \alpha \in \mathbb{Z}^n$$

and by Plancherel

$$\|f\|_{L^2[-\pi,\pi]^n} \cong \sum_{\alpha \in \mathbb{Z}^n} |\hat{f}(\alpha)|^2 .$$

Then $f \in H^k([-\pi,\pi]^n) \iff \sum_{\alpha \in \mathbb{Z}^n} (1+|\alpha|^2)^k |\hat{f}(\alpha)|^2 < \infty .$

$$\|f\|^2_{H^k(-\pi,\pi)^n} \cong \sum_{\alpha} (1+|\alpha|^2)^k |\hat{f}(\alpha)|^2$$