Ex 3

(a) Associativity : ✓

(b) $*$ is antilinear map : ✓

(c) Involution : ✓

(d) Homogenity :

$$\|\mu(\lambda,a)\|_{\tilde{A}} = \|(\mu\lambda,\mu a)\|_{\tilde{A}}$$

$$= \sup_{\|b\|_A=1} \|\mu\lambda b + \mu a b\|_A = |\mu| \cdot \sup_{\|b\|_A=1} \|\lambda b + ab\|_A$$

$$= |\mu| \cdot \|(\lambda,a)\|_{\tilde{A}}$$

(e) pos. def :

- $\|(\lambda,a)\| \geq 0$ clear
- $\|(\lambda,a)\| = 0$:

– If $\lambda = 0$ and $a \neq 0$ $\Rightarrow$ $0 = \|(0,a)\|$

$$= \sup_{\|b\|=1} \|ab\| \geq \|a \cdot \frac{a^*}{\|a^*\|}\| = \frac{1}{\|a^*\|} \|aa^*\| = \|a^*\| = \|a\|$$

$\Rightarrow$ $a = 0$ since $\|\cdot\|_A$ pos-def. $\Rightarrow$ $\lightning$ to $a \neq 0$

– So : if $\lambda = 0 \Rightarrow a = 0$

– Now $\lambda \neq 0$. In fact, w.l.o.g $\lambda = 1$ by part (a). Then

$$0 = \|(1,a)\| = \sup_{\|b\|=1} \|b + ab\|$$

$\Rightarrow$ $b + ab = 0$ $\quad \forall b \in A,\ \|b\| = 1$

$\Rightarrow$ $b + ab = 0$ $\quad \forall b \in A$

$\Rightarrow$ $b = -ab$ $\quad \forall b \in A$

$\Rightarrow$ $b^* = b^*(-a)$ $\quad \forall b \in A$

$\Rightarrow$ $b = b(-a^*)$ $\quad \forall b \in A$

– In part, $a^* = -aa^* = a$ $\Rightarrow$

$-a = -a^*$ is a unit $\lightning$

So: $\|(\lambda, a)\| = 0$ iff $\lambda = 0,\ a = 0$

(f) Δ- inequality :

$$\| (\lambda_1, a_1) + (\lambda_2, a_2) \|_{\tilde{A}} = \| (\lambda_1 + \lambda_2, a_1 + a_2) \|_{\tilde{A}}$$

$$= \sup_{\|b\|=1} \| (\lambda_1 + \lambda_2) b + (a_1 + a_2) b \|$$

$$\leq \sup_{\|b\|=1} (\| \lambda_1 b + a_1 b \| + \| \lambda_2 b + a_2 b \|)$$

$$\leq \sup_{\|b_1\| = \|b_2\| = 1} (\| \lambda_1 b_1 + a_1 b_1 \| + \| \lambda_2 b_2 + a_2 b_2 \|)$$

$$= \| (\lambda_1, a_1) \|_{\tilde{A}} + \| (\lambda_2, a_2) \|_{\tilde{A}}$$

(g) Sub-multiplicativity:

$$\|(\lambda_1, a_1)(\lambda_2, a_2)\|_{\hat{A}} = \|(\lambda_1\lambda_2,\ \lambda_1 a_2 + \lambda_2 a_1 + a_1 a_2)\|$$

$$= \sup_{\|b\|=1} \|\lambda_1\lambda_2 b + \lambda_1 a_2 b + \lambda_2 a_1 b + a_1 a_2 b\|$$

$$= \sup_{\|b\|=1} \|\lambda_1(\lambda_2 b + a_2 b) + a_1(\lambda_2 b + a_2 b)\|$$

$$\cancel{\le \sup_{\|b\|=1} \|\lambda_1 \|\lambda_2 b + a_2 b\| \cdot \frac{(\lambda_2 b + a_2 b)}{\|\lambda_2 b + a_2 b\|} + a_2 \|\lambda_2 b + a_2 b\|}$$

$$\le \sup_{\|b\|=1} \|\lambda_1 \cdot \|\lambda_2 b + a_2 b\| \cdot \frac{(\lambda_2 b + a_2 b)}{\|\lambda_2 b + a_2 b\|} + a_1 \|\lambda_2 b + a_2 b\| \cdot \frac{(\lambda_2 b + a_2 b)}{\|\lambda_2 b + a_2 b\|}\|$$

$$= \sup_{\|b\|=1} \|\lambda_2 b + a_2 b\| \cdot \|\lambda_1 \cdot \frac{(\lambda_2 b + a_2 b)}{\|\lambda_2 b + a_2 b\|} + a_1 \cdot \frac{(\lambda_2 b + a_2 b)}{\|\lambda_2 b + a_2 b\|}\|$$

$$\le \|(\lambda_2, a_2)\|_{\hat{A}} \cdot \|(\lambda_1, a_1)\|_{\hat{A}}$$

(i) C^*-property:

$$\|(\lambda,a)\|_{\tilde{A}}^2 \overset{(h)}{\leq} \|(\lambda,a)^*(\lambda,a)\|$$

$$\leq \|(\lambda,a)^*\| \cdot \|\lambda,a\|$$

$$= \|(\lambda,a)\|^2$$

$$\Rightarrow \quad \|(\lambda,a)\|_{\tilde{A}}^2 = \|(\lambda,a)^* \cdot (\lambda,a)\|$$

(h) $*$ is an isometry :

$$\|(\lambda,a)\|_{\hat{A}}^2 = \sup_{\|b\|=1} \|\lambda b + ab\|^2$$

$$= \sup_{\|b\|=1} \|(b^*\bar{\lambda} + b^*a^*)(\lambda b + ab)\|$$

$$\leq \sup_{\|b\|=1} \|b^*\| \cdot \|\bar{\lambda}\lambda b + \bar{\lambda}ab + a^*\lambda b + a^*ab\|$$

$$= \|(\lambda,a)^* \cdot (\lambda,a)\|_{\hat{A}}$$

$$\overset{(g)}{\leq} \|(\lambda,a)^*\|_{\hat{A}} \, \|(\lambda,a)\|_{\hat{A}}$$

$$\Rightarrow \quad \|(\lambda,a)\|_{\hat{A}} \leq \|(\lambda,a)^*\|_{\hat{A}}$$

$$\Rightarrow \quad \|(\lambda,a)\|_{\hat{A}} = \|(\lambda,a)^*\|_{\hat{A}}$$

by exchanging the roles of (λ,a) and $(\lambda,a)^*$

(j) $A \hookrightarrow \tilde{A}$ inclusion

- $\| (\lambda, 0) \| = \sup\limits_{\|b\|=1} \| \lambda b \| = \sup\limits_{\|b\|=1} |\lambda| \, \|b\| = |\lambda|$

- $\| (0, a) \|_{A} = \sup\limits_{\|b\|=1} \| ab \| \leq \|a\|$
- $\| (0, a) \| \geq \| a \cdot \frac{a^*}{\|a^*\|} \| = \|a\|$

$\Big\} \; \|(0,a)\|_{\hat{A}} = \|a\|$

(k) $\tilde{A}$ complete:

- $\delta := \inf\limits_{a \in A} \| (1, a) \| \geq 0$

- $\delta > 0$: If not $\Rightarrow \exists$ sequence $(a_n) \subseteq A$

s.t. $\|(1, a_n)\| \to 0$ for $n \to \infty$

$\Rightarrow$ $(1, a_n)_n$ convergent sequence

$\Rightarrow$ $(a_n)_n \subseteq A$ Cauchy sequence because

$$\|a_n - a_m\| = \|(0, a_n - a_m)\| = \|(1, a_n) - (1, a_m)\|$$

$\underset{\text{Complete}}{\overset{A}{\Rightarrow}}$ $(a_n)_n$ converges ; $a := \lim_n a_n$

- Now $\|(1, a_n)\| = \|(1, a) - (0, a - a_n)\|$

$$\geq \|(1,a)\| - \|(0, a - a_n)\|$$

$$= \underbrace{\|(1,a)\|}_{>0} - \underbrace{\|a - a_n\|}_{\to 0}$$

gives contradiction to $\|(1, a_n)\| \to 0$

$\Rightarrow$ $\delta > 0$

- $\lambda \in \mathbb{C} \Rightarrow$

$$|\lambda| \cdot \delta = \inf_{a \in A} \| (\lambda, \lambda a) \|$$

$$\leq \| (\lambda, c) \| \quad \forall c \in A$$

- (λ_n, a_n) Cauchy-sequence in $\hat{A}$

$$\Rightarrow \| (\lambda_n, a_n) - (\lambda_m, a_m) \| = \| (\lambda_n - \lambda_m, a_n - a_m) \|$$

$$\geq \delta \cdot |\lambda_n - \lambda_m| \quad \Rightarrow \quad |\lambda_n - \lambda_m| \leq \frac{1}{\delta} \| (\lambda_n, a_n) - (\lambda_m, a_m) \|$$

$\Rightarrow (\lambda_n)_n \subseteq \mathbb{C}$ Cauchy sequence ; $\lambda := \lim_n \lambda_n$

- $(a_n)_n \subseteq A$ also Cauchy-sequence :

$$\| a_n - a_m \| = \| (0, a_n - a_m) \| \cancel{= \| (\lambda_n, a_n)}$$

$$= \| (\lambda_n, a_n) - (\lambda_m, a_m) - (\lambda_n, 0) + (\lambda_m, 0) \|$$

$$\leq \underbrace{\| (\lambda_n, a_n) - (\lambda_m, a_m) \|}_{C.S} + \underbrace{|\lambda_n - \lambda_m|}_{C.S}$$

$\Rightarrow \quad \exists\, a \in A$ s.t. $\lim_n a_n = a$

- $\| (\lambda, a) - (\lambda_n, a_n) \|$

$$= \| (\lambda - \lambda_n, 0) + (0, a - a_n) \|$$

$$\le |\lambda - \lambda_n| + \| a - a_n \| \longrightarrow 0$$

$\Rightarrow \quad (\lambda_n, a_n) \longrightarrow (\lambda, a)$ in $\bar{A}$

$\Rightarrow \quad \bar{A}$ complete.