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Operator Algebras-A Tool Kit

Exercise Sheet 9

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Exercise 1: In the following write $\mathbf{k} = (k_1, \ldots, k_n)$ for elements of $\mathbb{Z}^n$ and think of them as multi-indices. In particular, write $e_i = (0, \ldots, 1, \ldots, 0)$ for the canonical basis of $\mathbb{Z}^n$ and $\mathbf{0} = (0, \ldots, 0)$ for its unit element. If $(A, \mathbb{T}^n, \alpha)$ is a C^*-dynamical system, then the space

$$A_{\mathbf{k}} := \{a \in A : \ (\forall z \in \mathbb{T}^n) \ \alpha(z).a = z^{\mathbf{k}} \cdot a\}$$

is called the *isotypic component* of $(A, \mathbb{T}^n, \alpha)$ corresponding to $\mathbf{k} \in \mathbb{Z}^n$. The space $A_{\mathbf{0}}$ is also called the *fixed point algebra* of the action $\alpha : \mathbb{T}^n \to A$.

(a) The group $\mathbb{T}^n$ acts on itself by left translation. Let

$$\alpha : \mathbb{T}^n \times C(\mathbb{T}^n) \to C(\mathbb{T}^n), \ \ \alpha(z, f)(w) := f(z^{-1}w)$$

be the corresponding action of $\mathbb{T}^n$ on the function space $C(\mathbb{T}^n)$. Compute the corresponding isotypic components $C(\mathbb{T}^n)_{\mathbf{k}}$.

(b) There is a continuous action α of $\mathbb{T}^n$ on A^n_θ by algebra automorphisms, which is on generators given by

$$\alpha(z).U^{\mathbf{k}} := z.U^{\mathbf{k}} := z^{\mathbf{k}} \cdot U^{\mathbf{k}} \ \text{ for } \ \mathbf{k} \in \mathbb{Z}^n.$$

Here $U^{\mathbf{k}} := U_1^{k_1} \cdots U_n^{k_n}$. Compute the corresponding isotypic components $(A^n_\theta)_{\mathbf{k}}$.

(c) The map

$$\alpha : \mathbb{T}^2 \times C^*(H) \to C^*(H), \ \ \alpha(z, w).U^k V^l W^m := z^k w^l \cdot U^k V^l W^m,$$

where $k, l, m \in \mathbb{Z}$, defines a continuous action of $\mathbb{T}^2$ on the group C^*-algebra of the discrete Heisenberg group $C^*(H)$ by algebra automorphisms. Compute the corresponding isotypic components $C^*(H)_{(k,l)}$.

Remark: The algebraic direct sum $\bigoplus_{\mathbf{k} \in \mathbb{Z}^n} A_{\mathbf{k}}$ of the isotypic components of a C^*-dynamical system $(A, \mathbb{T}^n, \alpha)$ is dense in A. In fact, this statement is a consequence of *Fourier theory.* For example any element $f \in C(\mathbb{T}^n)$ can be developed as a *Fourier series*:

$$f(z) \backsim \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}} \exp(2\pi i \mathbf{k} z),$$

where $\exp(2\pi i \mathbf{k} z) := \exp(2\pi i k_1 z_1) \cdots \exp(2\pi i k_n z_n)$. Note that the series converges uniformly to the function f only under certain conditions (say, if f is piecewise C^1).

Exercise 2: A C^*-dynamical system $(A, \mathbb{T}^n, \alpha)$ is called a *trivial noncommutative principal* $\mathbb{T}^n$-*bundle* if each isotypic component $A_{\mathbf{k}}$ contains an invertible element. Show that the C^*-dynamical systems $(C(\mathbb{T}^n), \mathbb{T}^n, \alpha)$, $(A^n_\theta, \mathbb{T}^n, \alpha)$ and $(C^*(H), \mathbb{T}^2, \alpha)$ from Exercise 1 provide examples of such trivial noncommutative principal torus-bundles.

Exercise 3 (On Projections):

(a) Show that equivalence $\sim$, unitary equivalence $\sim_u$ and homotopy $\sim_h$ actually define equivalence relations on the space $\mathcal{P}(A)$ of projections of a C^*-algebra A.

(b) Show that unitary equivalence $\sim_u$ implies equivalence $\sim$, i.e., show that $p \sim_u q \Rightarrow p \sim q$.

(c) Prove Lemma 6.1.7, i.e., show that that addition on $V(A)$ is well-defined and abelian.