3 November 2014

Operator Algebras-A Tool Kit

Exercise Sheet 8

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For the following exercises we recall that a (left-) Haar measure dg on a locally compact group G satisfies

$$\int_G f(xg)dg = \int_G f(g)dg, \text{ for all } f \in C_c(G,A) \text{ and } x \in G.$$

Here A can be any complete locally convex space, so in particular a C^*-algebra.

Exercise 1: Let (A, G, α) be a C^*-dynamical system and $f, h \in C_c(G, A)$. Verify the following statements:

(a) We have $f * h \in C_c(G, A)$ with $\operatorname{supp}(f * h) \subseteq \operatorname{supp}(f)\operatorname{supp}(h)$.

(b) The convolution product given by

$$(f * h)(x) := \int_G f(g)\alpha_g(h(g^{-1}x))dg$$

turns $C_c(G, A)$ into an associative algebra.

(c) We have $\|f * h\|_1 \leq \|f\|_1 \|h\|_1$.

(d) If G is *unimodular*, e.g. abelian or compact, then $f^*(g) := \alpha_g(f(g^{-1})^*)$ defines an involution on $C_c(G, A)$ with $\|f^*\|_1 = \|f\|_1$.

(e) The norm $\|f\| := \sup_\rho\{\|\rho(f)\|\}$ on $C_c(G, A)$ defines a C^*-norm.

Exercise 2: Let A be a unital C^*-algebra and G a discrete group. For $g \in G$ let v_g be the element of $C_c(G, A)$ which takes the value 1_A at g and 0 at the other elements of G.

(a) Describe the elements of $C_c(G, A)$.

(b) Prove Theorem V.5.7 in this context.

Exercise 3 (Stone–von Neumann): Suppose that G is a finite group with $|G| = n$. Then G acts on itself by left translation. Let $\alpha : G \times C(G) \to C(G)$, $\alpha(g, f)(x) := f(g^{-1}x)$ be the corresponding action of G on the function space $C(G)$. Prove that $C(G) \rtimes_\alpha G \cong M_n(\mathbb{C})$. Proceed as follows:

(a) Show that $(C(G), G, \alpha)$ defines a C^*-dynamical system.

(b) The Haar measure on G is given by the counting measure: $\int_G f(g)dg = \sum_{g\in G} f(g)$.

(c) Let $\pi : C(G) \to B(L^2(G))$ be given by pointwise multiplication: $(\pi(f).h)(x) := f(x) \cdot h(x)$.

(d) Let $u : G \to U(L^2(G))$ be given by the left-regular representation: $(u(g).h)(x) := h(g^{-1}x)$.

(e) Show that the pair (π, u) defines a covariant representation of $(C(G), G, \alpha)$ and let ρ be the corresponding representation of $C(G) \rtimes_\alpha G$.

(f) Let $G = \{g_i\}_{i=1}^n$ with $g_1 := e$. Then $L^2(G)$ is a n-dimensional Hilbert space with orthonormal basis $\{\delta_g\}_{g\in G}$, where δ_g is the function which is 1 at g and 0 at the other elements of G. In the following view operators on $L^2(G)$ as $n \times n$-matrices calculated with respect to $\{\delta_g\}_{g\in G}$.

(g) Show that if $f \in C(G, C(G)) \cong C(G \times G)$ and $h \in L^2(G)$, then

$$(\rho(f).h)(x) := \sum_{g\in G} f(g, x)h(g^{-1}x) = \sum_{g\in G} f(xg^{-1}, x)h(g).$$

(h) Conclude that $\rho(f)$ is given by the matrix M^f with (x, g)-th entry $M^f_{x,g} := f(xg^{-1}, x)$.

(i) If $M = (m_{x,g})_{x,g\in G}$ is any $n \times n$-matrix, then $f(x, g) := m_{g,x^{-1}g}$ satisfies

$$M^f_{x,g} = f(xg^{-1}, x) = m_{x,g}.$$

(j) Conclude that ρ is a surjective $*$-isomorphism of $C(G \times G)$ onto $M_n(\mathbb{C})$.

(k) Conclude that $C(G) \rtimes_\alpha G \cong M_n(\mathbb{C})$.

Remark: With substantially more effort one can actually prove that

$$C_0(G) \rtimes_\alpha G \cong \mathcal{K}(L^2(G))$$

holds for each locally compact group. Here $\mathcal{K}(L^2(G))$ denotes the C^*-algebra of compact operators on the Hilbert space $L^2(G)$.