OPERATOR ALGEBRAS–A TOOLKIT
EXERCISE SHEET 7

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1. GNS-representation of M_n Revisited

Let φ be a faithful state on M_n. Then φ is of the form $\varphi(x) = \mathrm{Tr}(\rho \cdot x)$ for some positive invertible matrix $\rho \in M_n$ (see exercise sheet 5). Show that the following representations are both the GNS-representation of M_n with respect to φ:

(1) The Hilbert space is $\mathcal{H} := M_n$ with the scalar product $\langle a, b\rangle := \mathrm{Tr}(b^* a)$ ($a, b \in M_n$), the representation is $\pi(x)a := xa$ ($x \in M_n, a \in M_n$), and the cyclic vector is

$$\xi_\varphi := \sqrt{\rho}.$$

(2) We fix a orthonormal basis $e_1, \dots, e_n \in \mathbb{C}^n$ of eigenvectors of ρ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n \geq 0$. Then the GNS-representation is given by the Hilbert space $\mathcal{H} := \mathbb{C}^n \otimes \mathbb{C}^n$, the representation $\pi(x) := x \otimes \mathbb{1}$ ($x \in M_n$), and the cyclic vector

$$\xi_\varphi := \sum_{i=1}^{n} \sqrt{\lambda_i}\, e_i \otimes e_i.$$

Notice that only the cyclic vector (not the Hilbert space or the representation) depends on the choice of the state φ.

2. The CAR-Algebra

(1) Consider the CAR-algebra over a singleton index set, i. e., consider an algebra $\mathcal{A}$ generated by a single element a satisfying

$$a^2 = 0, \qquad\qquad a^* a + a a^* = \mathbb{1}.$$

Verify that $p_1 := a^* a$ and $p_2 := a a^*$ are computing projections. Show that the elements $a, a^*, a^* a, a a^*$ are a *linear* basis of $\mathcal{A}$; in particular, $\mathcal{A}$ is 4-dimensional. Show that $\mathcal{A}$ is isomorphic to M_2.

(2) Now consider the index set $I = \{1, \dots, N\}$. For $i \in I$ we define elements of the N-fold tensor product $M_2 \otimes \ldots \otimes M_2$ by

$$a_i := \sigma_z \otimes \ldots \otimes \sigma_z \otimes a \otimes \mathbb{1} \otimes \ldots \mathbb{1}$$

with a in the i-th tensor factor, where $\sigma_z := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $a := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Show that the elements a_i ($i \in I$) satisfy the CAR relation.

(3) Consider the Hilbert space $\mathcal{H} := \mathbb{C}^N$. For $\xi = (\xi_1, \dots, \xi_N) \in \mathcal{H}$ put

$$a^*(\xi) := \sum_{i=1}^{N} \xi_i \cdot a_i^*, \qquad\qquad a(\xi) := \big(a^*(\xi)\big)^*.$$

Show that for all $\xi, \eta \in \mathcal{H}$ we have

$$a(\xi)\, a(\eta) + a(\eta)\, a(\xi) = 0,$$
$$a^*(\xi)\, a(\eta) + a(\eta)\, a^*(\xi) = \langle \xi, \eta\rangle\, \mathbb{1}.$$

(4) Show that the map $\mathcal{H} \to \mathcal{A}(I)$, $\xi \mapsto a^*(\xi)$ is a linear isometry.
Hint: You may revisit the first part of this exercise. Alternatively, you may investigate $\|a^*(\xi)\|^4$.

(5) Let U be a unitary $N \times N$-matrix (i. e. a unitary operator on $\mathcal{H}$). Show that there is a unique *-automorphism

$$\varphi_U : \mathcal{A}(I) \to \mathcal{A}(I), \quad a^*(\xi) \mapsto a^*(U\xi) \qquad\qquad (\xi \in \mathcal{H}).$$

Hint: You may proceed similar to Exercise 3, part 2.

3. The Cuntz algebra

Suppose that s_1, s_2 satisfy the Cuntz relation. Show the following:

(1) $s_1^* s_2 = 0 = s_2^* s_1$.

(2) Let $U := \left(\begin{smallmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{smallmatrix}\right)$ be a unitary 2×2-matrix. Show that the elements

$$\tilde{s}_1 := u_{11}\, s_1 + u_{12}\, s_2,$$
$$\tilde{s}_2 := u_{21}\, s_1 + u_{22}\, s_2$$

again satisfy the Cuntz relation. Conclude that there is a unique *-automorphism

$$\phi_U : \mathcal{O}_2 \to \mathcal{O}_2, \quad s_i \mapsto \tilde{s}_i \qquad\qquad (i = 1, 2)$$

(3) The following elements also satisfy the Cuntz relations:

$$s_1 s_1,\ s_1 s_2,\ s_2 s_1,\ s_2 s_2.$$

Conclude that $\mathcal{O}_4$ is a unital subalgebra of $\mathcal{O}_2$.

(4) The following elements generate a C*-subalgebra that is isomorphic to M_2:

$$s_1 s_1^*,\ s_1 s_2^*,\ s_2 s_1^*,\ s_2 s_2^*$$

We denote this subalgebra by $\mathcal{A}_1$.
Hint: You may follow similar lines as in the first part of Exercise 2.

(5) The following elements also generate a C*-subalgebra isomorphic to M_2:

$$s_1^2 (s_1^2)^*,\ (s_1^2)(s_2^2)^*,\ (s_2^2)(s_1^2)^*,\ (s_2^2)(s_2^2)^*$$

We denote this subalgebra by $\mathcal{A}_2$.

(6) The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ commute elementwise. Conclude that the subalgebra jointly generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ is isomorphic to $M_2 \otimes M_2$.

(7) Find a subalgebra of $\mathcal{O}_2$ that is isomorphic to the k-fold tensor product $M_2 \otimes \ldots \otimes M_2$.