OPERATOR ALGEBRAS–A TOOLKIT
EXERCISE SHEET 6

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1. Commutants are C*-algebras

Let S be a subset of $\mathcal{B}(\mathcal{H})$ with $S = S^*$. Show that the commutant S' is a closed *-subalgebra and hence C*-algebra.

2. Degenerate Representations

Let $(\pi, \mathcal{H})$ be a representation of a C*-algebra $\mathcal{A}$.

(1) Show that the set of all vectors $\xi \in \mathcal{H}$ with $\pi(\mathcal{A})\xi = 0$ is a closed linear subspace of $\mathcal{H}$.

(2) Conclude that $(\pi, \mathcal{H})$ can be uniquely written as a direct sum $\pi = \pi_1 \oplus \pi_2$ where π_1 is the trivial zero representation and π_2 is non-degenerate.

3. Construction of the GNS-Representation

In this exercise we fill some gaps from the lecture in the construction of the GNS-representation. Let φ be a state on a unital C*-algebra $\mathcal{A}$. Review your notes and show the following:

(1) The set $J := \{x \in \mathcal{A} \mid \varphi(x^*x) = 0\}$ is a left ideal in $\mathcal{A}$.

(2) The map $x \mapsto \pi_\varphi(x)$ is indeed a representation.

4. Properties of the GNS-Representation

Let $\mathcal{A}$ be a unital C*-algebra, φ a state on $\mathcal{A}$, and $\pi_\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\varphi)$ the corresponding GNS-construction. Show the following:

(1) φ is faithful if and only if ξ_φ is separating for $\pi_\varphi(\mathcal{A})$.

(2) If φ is faithful then π_φ is faithful.
Warning: The converse does not hold!

(3) φ is pure if and only if π_φ is irreducible.

5. GNS-Representation Concretely

Consider the algebra $\mathcal{A} = M_n$ of $n \times n$-matrices. For this exercise you may use the fact that M_n is *simple*, i. e., there are no two-sided ideals $J \subsetneq M_n$ except $J = \{0\}$ and $J = M_n$.

(1) First consider the state on M_n given by $\tau(x) := \frac{1}{n}\operatorname{Tr}(x)$, $x \in M_n$. Show that the GNS-representation of τ is given by:
- the Hilbert space $\mathcal{H} = M_n$ with the scalar product
$$\langle x, y\rangle = \tfrac{1}{n}\operatorname{Tr}(y^*x),$$
- the representation π given by $\pi(x)y = xy$.

(2) Now consider an arbitrary faithful state φ on M_n. By the previous exercise sheet it is given by $\varphi(x) = \operatorname{Tr}(\rho \cdot x)$, $x \in M_n$, for some (strictly) positive definite matrix $\rho \in M_n$. Determine the GNS-representation of φ.

(3) Now consider an arbitrary (non-faithful) state φ on M_n. Determine its GNS-representation.