OPERATOR ALGEBRAS–A TOOLKIT
EXERCISE SHEET 5

DR. KAY SCHWIEGER, DR. STEFAN WAGNER

1. States and the Norm

Show that for an element $x = x^*$ of a unital C*-algebra we have

$$\|x\| = \sup_{\varphi \text{ state}} |\varphi(x)| = \sup_{\varphi \text{ pure state}} |\varphi(x)|.$$

Hint: Use the Hahn-Banach extension theorem for functionals.

2. Projections

An element p is called an (orthogonal) projection if $p^* = p = p^2$.

(1) Show that for a self-adjoint element p are equivalent
  (a) p is a projection.
  (b) p has spectrum $\{0, 1\}$.
  Conclude that projections are positive elements.

(2) What are projections in $\mathcal{B}(\mathcal{H})$ in terms of $\mathcal{H}$? What are projections in $\mathcal{C}(X)$ in terms of X?

(3) Show that for two projections p, q the following are equivalent:
  (a) $p \geq q$.
  (b) $pq = q$.
  (c) $p - q$ is a projection.
  What do these conditions mean for projections in $\mathcal{B}(\mathcal{H})$?

3. Functionals on M_n

Show the following for the C*-algebra $\mathcal{A} = M_n$.

(1) The trace $\mathrm{Tr} : M_n \to \mathbb{C}$ is a faithful[1] positive functional.

(2) Every functional φ on M_n is of the form

$$\varphi(x) = \mathrm{Tr}(\rho \cdot x)$$

  for some unique matrix $\rho \in M_n$. The matrix ρ is called *density* of φ.

(3) φ is self-adjoint if and only if ρ is self-adjoint.
  φ is positive if and only if ρ is a positive semidefinite.

4. Positivity in M_2

(1) Consider an arbitrary self-adjoint matrix in M_2:

$$\begin{pmatrix} a+z & x+iy \\ x-iy & a-z \end{pmatrix}$$

  with $a, x, y, z \in \mathbb{R}$. When is this matrix positive? Find a characterization in terms of the coefficients a, x, y, z?

---

[1] A positive functional φ is called *faithful* if $\varphi(x^*x) > 0$ whenever $x \neq 0$.

(2) Use the results of Exercise 3. Show that the set of states on M_2 is up to a linear transformation given by the 3-dimensional unit ball

$$B := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}.$$

and the set of pure states is given by the 2-sphere

$$S := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The sphere S with this interpretation as states on M_2 is also called the *Bloch sphere*.

5. A New C*-algebra?

Let $\mathcal{A}$ be a unital C*-algebra and a an element with

$$a^2 = 0, \qquad a^*a + aa^* = \mathbb{1}$$

Determine the C*-algebra $C(a, \mathbb{1})$ generated by a and $\mathbb{1}$.