OPERATOR ALGEBRAS–A TOOLKIT
EXERCISE SHEET 4

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1. Unitary Elements as Exponentials

(1) Show that for every self-adjoint h the element $u := e^{ih}$ is unitary.
(2) Show the almost converse: Every unitary element u is of the form $u = e^{ih}$ for some self-adjoint element h if $\sigma(u)$ is not the full circle. What happens if $\sigma(u)$ is the full circle?

2. Properties of Functional Calculus

Let $\mathcal{A}$ be a unital C*-algebra and $x \in \mathcal{A}$. Let $f : \sigma(x) \to \mathbb{C}$ and $g : \mathbb{C} \to \mathbb{C}$ be continuous functions. Show the following:

(1) For a polynomial function $p(z) = \sum_n c_n z^n$: $\quad p(x) = \sum_n c_n x^n$.
(2) $\sigma\big(f(x)\big) = f\big(\sigma(x)\big)$.
(3) $\|f(x)\| = \sup_{\lambda \in \sigma(x)} |f(\lambda)|$.
(4) $g\big(f(x)\big) = (g \circ f)(x)$.

The rest of this exercise is optional: Let $f_n, f : [a, b] \to \mathbb{R}$ be continuous and $x_n, x \in \mathcal{A}$ be self-adjoint elements with spectrum in $[a, b]$.

(5) Show the implication

$$f_n \to f \text{ uniformly, } x_n \to x \qquad \Longrightarrow \qquad f_n(x_n) \to f(x)$$

Hint: Use the Stone-Weierstrass Theorem. Maybe try it first for the special cases $f_n = f$ or $x_n = x$.

3. Polar Decomposition

(1) Show that every invertible element x is of the form $x = uh$ with a unitary element u and a positive element h in the C*-algebra.

This decomposition (and variants) are called *polar decomposition*.

(2) Show that the elements u and h are unique.
(3) Compute the polar decomposition explicitly for

$$x = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}.$$

(4) Do we also have $x = hu$ in general?

4. Counterexamples for Positivity

4.1. Multiplication is Not Positive.

(1) Show that for commuting elements x, y we have the implication

$$x, y \geq 0 \qquad \Longrightarrow \qquad x \cdot y \geq 0.$$

(2) Show that if this implication holds for all elements $x, y \geq 0$ of a C*-algebra $\mathcal{A}$, then $\mathcal{A}$ is commutative.

4.2. Squaring is Not Positive.

(1) Consider the matrices $x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $y = x + \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Show that $0 \leq x \leq y$ but $x^2 \nleq y^2$.

(2) Show that if the implication

$$0 \leq x \leq y \qquad \Longrightarrow \qquad 0 \leq x^2 \leq y^2$$

holds for all elements x, y of a C*-algebra $\mathcal{A}$, then $\mathcal{A}$ is commutative.

5. C*-Algebras of Sequences

We consider the natural numbers $\mathbb{N}$ and write $c_0 := \mathcal{C}_0(\mathbb{N})$.

(1) Convince yourself that c_0 is the space of all sequences converging to zero.
(2) This part is optional and you may skip it: Find an closed *-ideal of c_0 that is not maximal. Characterize all closed *-ideals of c_0.
(3) What is the unitalization of c_0? Write it as a space of sequences and make the character φ with $\ker \varphi = c_0$ explicit.
(4) Show that the set of all bounded sequences

$$\ell^\infty := \left\{ x = (x_n)_{n \in \mathbb{N}} : \sup_{n \in \mathbb{N}} |x_n| < \infty \right\}$$

equipped with the norm $\|x\| := \sup_n |x_n|$ is a C*-algebra.
(5) Show that ℓ^∞ contains c_0 as an ideal but not as a maximal idea.

The rest of this exercise is optional (but interesting):

(6) Since ℓ^∞ is commutative we have $\ell^\infty = \mathcal{C}(X)$ for some compact space X. Show that $\mathbb{N}$ is a subset of X.
(7) Show that $\mathbb{N}$ is dense in X.
(8) Show that this space X is totally disconnected.

The space X is called the *Stone-Ĉech compactification*, usually denoted by $\beta\mathbb{N}$. It is in fact the largest compact space into which $\mathbb{N}$ can be densely embedded.