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Operator Algebras-A Tool Kit

Exercise Sheet 3

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Exercise 1: Let A be a Banach space. Show that the weak-*-topology on A^* is Hausdorff, i.e., show that two distinct continuous functionals have disjoint neighbourhoods with respect to the topology of pointwise convergence.

To each locally compact Hausdorff space we can associate the commutative C*-algebra $C_0(X)$. Conversely, to each commutative C*-algebra A we can associate the locally compact Hausdorff space Γ_A (Gelfand spectrum). The goal of the following exercise is to show that the assignments $C_0(\cdot)$ and $\Gamma_{(\cdot)}$ are inverse to each other (up to *-isomorphy and homeomorphy, respectively). In fact, Theorem II.19 says that $C_0(\Gamma_A)$ is *-isomorphic to A, so it reamins to show the following:

Exercise 2: Let X be a locally compact Hausdorff space. Show that the map

$$\Phi : X \to \Gamma_{C_0(X)}, \;\; x \mapsto \delta_x$$

is a homeomorphism. Proceed as follows:

(a) Show that Φ is injective and continuous. Hint: Use Urysohn's Lemma for the injectivity.

In the following assume that X is compact:

(b) Assume that there exists another character $\delta \in \Gamma_{C(X)}$ with $\delta \neq \delta_x$ for all $x \in X$. Show that $\{\delta_x : x \in X\}$ and δ are disjoint and compact subsets of $\Gamma_{C(X)}$.

(c) Choose a continuous function

$$F : \Gamma_{C(X)} \to \mathbb{C} \text{ such that } \begin{cases} F(\delta_x) = 0 & \text{for all } x \in X \\ F(\delta) = 1. \end{cases}$$

Use Theorem II.19 to show that there exists $f \in C(X)$ with $F = \widehat{f}$.

(d) Show that $f = 0$ and thus $F = 0$ in contradiction to $F(\delta) = 1$. Conclude that Φ is continuous and bijective.

(e) Show that Φ is a homeomorphism. Hint: Use a well-known Theorem from Topology.

Now back to locally compact X:

(f) Prove the statement for locally compact X by using the previous results and the one-point compactification $\widehat{X}$ of X.

Exercise 3: A map $g : X \to Y$ between locally compact Hausdorff spaces X and Y is called *proper* if the preimage of compact sets are compact. Prove the following statements:

(a) If $g : X \to Y$ is continuous and proper, then the map

$$\pi_g : C_0(Y) \to C_0(X), \quad f \mapsto f \circ g$$

is a *-homomorphism.

(b) Conversely, if $\pi : A_2 \to A_1$ is a *-morphism between C*-algebras A_1 and A_2, then the map

$$g_\pi : \Gamma_{A_1} \to \Gamma_{A_2}, \quad \varphi \mapsto \varphi \circ \pi$$

is continuous and proper.

(c) The assigments $g \mapsto \pi_g$ and $\pi \mapsto g_\pi$ are invers to each other.

Exercise 4 (A Dictionary): Complete the following dictionary by filling in the corresponding algebraic counterparts:

Topology	Algebra
locally compact space Hausdorff space	
compact Hausdorff space	
one-point compactification	
continuous and proper map	
homeomorphism	
points	