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# Operator Algebras-A Tool Kit

## Exercise Sheet 2

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**University of Helsinki, Fall 2014**

**Exercise 1 (Gelfand–Mazur):** Let $A$ be a unital C*-algebra. Show that if each $0 \neq a \in A$ is invertible, then $A = \mathbb{C} \cdot 1_A \cong \mathbb{C}$.

**Definition (Ideals in C*-Algebras):**

(a) A (*two-sided*) *ideal* $J$ of a C*-algebra is a vector subspace satisfying

$$x \in J, a \in A \Rightarrow xa \in J \text{ and } ax \in J.$$

If $J \neq A$, then $J$ is called a *proper ideal.*

(b) A *maximal ideal* $J$ is a proper ideal satisfying

$$J \subsetneq \tilde{J}, \tilde{J} \text{ ideal} \Rightarrow \tilde{J} = A.$$

**Remark (Ideals are automatically *-invariant):** We will see later in the course that ideals of C*-algebra are automatically closed under the *-operation, i.e., if $J$ is an ideal of a C*-algebra $A$, then $J^* = J$. In particular, closed ideals are C*-algebras in its own right. We will use this fact in the following exercises.

**Exercise 2:** Let $A$ be a C*-algebra. Show that:

(a) The closure of an ideal is again an ideal.

(b) If $J$ is a closed ideal, then $A/J$ equipped with the norm $\|[a]\|_{A/J} := \inf_{x \in J} \|a + x\|$ satisfies the conditions 1)-5) of Definition I.1. Here, $[a] \in A/J$ denotes the class of an element $a \in A$.

**Remark ($A/J$ as C*-Algebra):** With a little more refined methods it is actually possible to show that the quotient $A/J$ equipped with the norm of exercise 2(b) carries the structure of a C*-algebra. In fact, it only remains to verify the C*-property. We will prove that fact later in the course and use the result for now in the following exercises.

**Exercise 3:** Let $A$ be a unital C*-algebra. Show that:

(a) A proper ideal can not be dense in $A$. Hint: Use Proposition I.12.

(b) Maximal ideals are closed.

(c) Every proper ideal is contained in a maximal ideal. Hint: Use Zorns Lemma.

(d) If $J$ is a proper closed ideal, then the C*-algebra $A/J$ is unital.

**Exercise 4:** Let $A$ be a unital commutative C*-algebra. Show that if $J$ is a maximal ideal, then $A/J$ is one-dimensional.

Hint: Verify the following statements:

(a) The quotient $A/J$ is a commutative unital C*-algebra.

(b) In view of Exercise 1 (Gelfand-Mazur), it is enough to show that

$$\forall x \in A \backslash J \ \ \exists y \in A \ \text{ satisfying } \ [xy] = [1_A].$$

(c) For $x \in A \backslash J$ define

$$J_x := \{xa + b : \ a \in A, b \in J\}.$$

Show that $J$ is an ideal satisfying $J \subseteq J_x$.